MATH 2E03

SOLUTIONS TO HOMEWORK ASSIGNMENT #2

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1) a) ~~[illegible]~~ Accumulated value after T years

$$P(T) = P_0(1+r)^T - \left[(1+r)^T - 1\right]\frac{x}{r} \quad [\text{cf. } (3.8)]$$

Same, but compounding m times a year:

$$P_m(T) = P_0\left(1+\frac{r}{m}\right)^{mT} - \left[\left(1+\frac{r}{m}\right)^{mT} - 1\right]\frac{xm}{r}$$

$P_m(T) = 0$ — find T

$$P_0\left(1+\frac{r}{m}\right)^{mT} - \left[\left(1+\frac{r}{m}\right)^{mT} - 1\right]\frac{xm}{r} = 0$$

$$\left(1+\frac{r}{m}\right)^{mT}\left[P_0 - \frac{xm}{r}\right] = -\frac{xm}{r} \quad \Big/ \cdot (-1)$$

assume $\frac{xm}{r} > P_0$ and take $\ln$ of both sides

$$mT \ln\left(1+\frac{r}{m}\right) + \ln\left(\frac{xm}{r} - P_0\right) = \ln\frac{xm}{r}$$

Hence:

$$T = \frac{\ln\frac{xm}{r} - \ln\left(\frac{xm}{r} - P_0\right)}{m\ln\left(1+\frac{r}{m}\right)} = \frac{\ln\left(\frac{xm}{xm - P_0 r}\right)}{m\ln\left(1+\frac{r}{m}\right)}$$

monthly compounding: $m = 12$

$$T = \frac{\ln\left(\frac{12x}{12x - P_0 r}\right)}{12\ln\left(1+\frac{r}{12}\right)}$$

1b) Continuous compounding

Accumulated debt:

after T years: $P(T) = P_0 e^{gT} - \frac{x}{g}(e^{gT} - 1)$

$P(T) = 0 \Rightarrow e^{gT}(P_0 - \frac{x}{g}) = -\frac{x}{g} \quad / \cdot (-1)$ [cf. Section 3.2.3]

Assume $\frac{x}{g} > P_0$ and take ln of both sides

$$gT + \ln\left(\frac{x}{g} - P_0\right) = \ln \frac{x}{g}$$

$$T = \frac{\ln \frac{x}{g} - \ln\left(\frac{x}{g} - P_0\right)}{g} = \frac{\ln\left(\frac{x}{x - P_0 g}\right)}{g}$$

2)

$$\left.\begin{aligned} P(0) &= P(1)\, e^{-g_1(1-0)} \\ P(1) &= P(2)\, e^{-g_2(2-1)} \\ P(2) &= P(3)\, e^{-g_3(3-2)} \end{aligned}\right\} \Rightarrow P(0) = P(3)\, e^{-g_1} e^{-g_2} e^{-g_3} = P(3)\, e^{-(g_1+g_2+g_3)}$$

$P(3) = \$\, 10^4$

$P(0) = PV$ (the present value)

$$PV = \$\, 10^4 e^{-(0.04 + 0.06 + 0.05)} = \$\, 10^4 \cdot e^{-0.15} \cong \underline{\$\, 8607}$$

3)

$\frac{S'(t)}{S(t)} = -h(t)$, where the hazard rate function $h(t) = \begin{cases} \lambda > 0, & t \in [0, 50] \\ \lambda + d(t-50), & t \in (50, \infty) \end{cases}$

Integrate the equation for the survivor function

$$\int_{t_0}^{t} \frac{S'(\tau)}{S(\tau)}\, d\tau = \ln S(\tau)\Big|_{\tau=t_0}^{\tau=t} = \ln \frac{S(t)}{S(t_0)} = -\int_{t_0}^{t} h(\tau)\, d\tau$$

Hence: $\underline{S(t) = S(t_0) \exp\left[-\int_{t_0}^{t} h(\tau)\, d\tau\right]}$

- for $t \in [0, 50]$

$t_0 = 0 \Rightarrow S(0) = 1$

$$S(t) = 1 \cdot \exp\left[-\lambda \int_0^t d\tau\right] = \underline{e^{-\lambda t}}$$

- for $t \in (50, \infty)$

$t_0 = 50$, $S(50) = \exp(-50\lambda)$

$$S(t) = S(50) \exp\left[-\int_{50}^{t} \lambda + \alpha(\tau - 50)\, d\tau\right]$$

$$= \exp(-50\lambda) \exp\left[-(\lambda - \alpha 50)(t-50) - \alpha \frac{\tau^2}{2}\Big|_{50}^{t}\right]$$

$$= \exp(-50\lambda) \exp\left[-(\lambda - 50\alpha)t + 50\lambda - 2500\alpha - \alpha\frac{t^2}{2} + 1250\alpha\right]$$

$$= \exp\left[-\alpha\frac{t^2}{2} - (\lambda - 50\alpha)t - 1250\alpha\right]$$

Thus:

$$S(t) = \begin{cases} \exp(-\lambda t), & t \in [0, 50] \\ \exp\left[-\alpha\frac{t^2}{2} - (\lambda - 50\alpha)t - 1250\alpha\right], & t \in (50, \infty) \end{cases}$$

Note the function $S(t)$ is continuous at $t = 50$ as it should be.

4) At the lecture we chose t, E and ρ_0 as the primary quantities; now let's choose t, E and P_0

$$[P_0] = L_1 L_2^{-1} L_3^{-2} = \frac{kg}{m\,s^2}$$

$$[r] = [t]^\alpha [E]^\beta [P_0]^\gamma$$

$$\Downarrow$$

$$[L_2] = L_3^\alpha (L_1 L_2^2 L_3^{-2})^\beta (L_1 L_2^{-1} L_3^{-2})^\gamma$$

$$\Downarrow$$

$$\begin{bmatrix} 0 & 1 & 1 \\ 0 & 2 & -1 \\ 1 & -2 & -2 \end{bmatrix} \begin{bmatrix} \alpha \\ \beta \\ \gamma \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \Rightarrow \beta = 1/3,\ \gamma = -1/3 \Rightarrow \alpha = 0$$

Thus: $[r] = D = E^\beta P_0^\gamma = \left(\frac{E}{P_0}\right)^{1/3}$

and $\Pi = r\left(\frac{P_0}{E}\right)^{1/3}$

The secondary quantity ρ_0: $[\rho_0] = \frac{kg}{m^3} = \underline{L_1 L_2^{-3}}$

$$[\rho_0] = [t]^\alpha [E]^\beta [P_0]^\gamma$$

$$\Downarrow$$

$$L_1 L_2^{-3} = L_3^\alpha (L_1 L_2^2 L_3^{-2})^\beta (L_1 L_2^{-1} L_3^{-2})^\gamma$$

$$\Downarrow$$

$$\begin{bmatrix} 0 & 1 & 1 \\ 0 & 2 & -1 \\ 1 & -2 & -2 \end{bmatrix} \begin{bmatrix} \alpha \\ \beta \\ \gamma \end{bmatrix} = \begin{bmatrix} 1 \\ -3 \\ 0 \end{bmatrix} \Rightarrow \beta = -\frac{2}{3},\ \gamma = \frac{5}{3} \Rightarrow \alpha = 2$$

Thus $[\rho_0] = D_1 = t^2 E^{-\frac{2}{3}} P_0^{\frac{5}{3}} = t^2 \left(\frac{P_0^5}{E^2}\right)^{1/3}$

and $\Pi_1 = \rho_0 t^{-2} \left(\frac{E^2}{P_0^5}\right)^{1/3}$

Therefore $r = DF(\Pi_1) = \left(\frac{E}{p_o}\right)^{1/3} F\left(\rho_o t^{-2}\left(\frac{E^2}{p_o^5}\right)^{1/3}\right)$

For large times & small energies

$$r \cong \left(\frac{E}{p_o}\right)^{1/3} F(0) \cong \left(\frac{E}{p_o}\right)^{1/3}$$

5) Dimension of force: $[K] = \frac{kg\, m}{s^2} = L_1 \cdot L_2 \cdot L_3^{-2}$

The primary quantities:

$[\rho] = \frac{kg}{m^3} = L_1 \cdot L_2^{-3}$, $[A] = m^2 = L_2^2$, $[u] = \frac{m}{s} = L_2 \cdot L_3^{-1}$

Thus: $[K] = [\rho]^\alpha [A]^\beta [u]^\gamma$

$$\Downarrow$$

$$L_1 \cdot L_2 \cdot L_3^{-2} = (L_1 L_2^{-3})^\alpha (L_2^2)^\beta (L_2 \cdot L_3^{-1})^\gamma$$

$$\begin{bmatrix} 1 & 0 & 0 \\ -3 & 2 & 1 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} \alpha \\ \beta \\ \gamma \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix} \quad \left.\begin{matrix} \Rightarrow \alpha = 1 \\ \\ \Rightarrow \gamma = 2 \end{matrix}\right\} \Rightarrow \beta = 1$$

Therefore $[K] = D = \rho A u^2$

and $\Pi = K \rho^{-1} A^{-1} u^{-2}$

No secondary variables $\Rightarrow$ Π reduces to a constant c

$$F = D\Pi = c\rho A u^2$$