MATH 2E03 – TEST #2

SOLUTIONS

1) $u = g(V, g)$

Dimensions:

$$\left.\begin{aligned}[u] &= \frac{m}{s} = L_2 L_3^{-1} \\ [V] &= m^3 = L_2{}^3 \\ [g] &= \frac{m}{s^2} = L_2 L_3^{-2}\end{aligned}\right\} \Rightarrow$$

two fundamental dimensional units L_2, L_3 & two primary variables V and g

$$[u] = [V]^\alpha [g]^\beta \Rightarrow L_2 L_3^{-1} = (L_2^3)^\alpha (L_2 L_3^{-2})^\beta$$

$$\Rightarrow \left.\begin{bmatrix} 3 & 1 \\ 0 & -2 \end{bmatrix} \begin{bmatrix} \alpha \\ \beta \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \end{bmatrix} \Rightarrow \beta = \tfrac{1}{2}\right\} \Rightarrow \alpha = \tfrac{1}{6}$$

Thus: $[u] = D = V^{1/6} g^{1/2}$

and $\Pi = \dfrac{u}{D} = \dfrac{u}{V^{1/6} g^{1/2}}$

No secondary variables $\Rightarrow$ Π reduces to a <u>constant</u> C $\Rightarrow$ $\underline{\underline{u = C V^{1/6} g^{1/2}}}$

2) $\ddot{x} + \omega_0^2 x + \beta x^3 = 0$, $\omega, \beta \in \mathbb{R}$, $\beta < 0$

a) Compute linearization around $(x, \dot{x}) = (0, 0)$

$$\underbrace{\begin{aligned}\dot{x} &= y && = g(x,y) \\ \dot{y} &= -\omega_0^2 x - \beta x^3 && = h(x,y)\end{aligned}}_{\text{equivalent system of first-order ODEs}}$$

$$\frac{d}{dt}\begin{bmatrix} X \\ Y \end{bmatrix} = \begin{bmatrix} g_x & g_y \\ h_x & h_y \end{bmatrix}\Bigg|_{(0,0)} \begin{bmatrix} X \\ Y \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & 1 \\ -\omega_0^2 & 0 \end{bmatrix}}_{A} \begin{bmatrix} X \\ Y \end{bmatrix}$$

X, Y – perturbation variables

Finding eigenvalues of A

$$\operatorname{Det}(A-\lambda I) = \begin{vmatrix} -\lambda & 1 \\ -\omega_0^2 & -\lambda \end{vmatrix} = \lambda^2+\omega_0^2 = 0 \Rightarrow \underline{\lambda = \pm i\omega_0}$$

Thus, we have centre equilibrium which is ~~the~~ linearly stable, but structurally unstable

b) Other equilibrium solutions

$g(x,y) = 0$ $y = 0$

$h(x,y) = 0$ $-\omega_0^2 x - \beta x^3 = 0$

$$-x(\omega_0^2+\beta x^2) = 0 \Rightarrow x = \frac{\omega_0}{\sqrt{-\beta}}$$

~~the~~ if now $\beta > 0$ $x = \dfrac{\omega_0}{i\sqrt{\beta}} = -i\dfrac{\omega_0}{\sqrt{\beta}}$

Not a real solution anymore!

3) $\begin{cases} \dfrac{d^2y}{dx^2} = h \\ y(0) = 0 \\ y'(1) = h^2 \end{cases}$

General solution

$$y(x) = \frac{1}{2}hx^2 + c_1 x + c_2$$

$$y'(x) = hx + c$$

Finding constants c_1 and c_2

$$y(0) = c_2 = 0 \Rightarrow c_2 = 0$$

$$y'(1) = h\cdot 1 + c_2 = h^2 \Rightarrow c_1 = h^2 - h$$

Thus:

$$y(x) = \frac{1}{2}hx^2 + h(h-1)x$$

The point of maximum deflection

$$y'(x) = hx + h(h-1) = 0 \Rightarrow x_{max} = \underline{1-h}$$

The maximum deflection

$$y_{max} = y(x_{max}) = \frac{1}{2}h(1-h)^2 + h(h-1)(1-h)$$
$$= \frac{1}{2}h(1-h)^2 - h(1-h)^2 = -\frac{1}{2}h(1-h)^2 = -\frac{1}{2}h + h^2 - \frac{1}{2}h^3$$

Finding the <u>optimum</u> h:

$$\frac{d}{dh} y_{max} = -\frac{1}{2} + 2h - \frac{3}{2}h^2 = 0$$

Equivalently: $3h^2 - 4h + 1 = 0$

$$\Delta = (-4)^2 - 4 \cdot 3 \cdot 1 = 4, \quad h_1 = \frac{4+\sqrt{4}}{6} = 1, \quad h_2 = \frac{4-\sqrt{4}}{6} = \frac{1}{3}$$

$$y_{max}(h=1) = 0$$

$$y_{max}\left(h = \frac{1}{3}\right) = -\frac{1}{2}\frac{1}{3}\left(1 - \frac{1}{3}\right)^2 = -\frac{1}{6}\frac{4}{9} = -\frac{2}{27}$$

Thus $h = \frac{1}{3}$ gives the largest deflection.