HOMEWORK #3

Due: March 4 (Tuesday) by midnight

Instructions:

- The assignment consists of *three* questions, worth 3, 2, and 5 points.
- Submit your assignment *electronically* (via Email) to the address `math2t03@math.mcmaster.ca`; hardcopy submissions will not be accepted.
- It is obligatory to use the MATLAB template file available at `http://www.math.mcmaster.ca/~bprotas/MATH2T03/template.m` (see also the link in the "Computer Programs" section of the course website on the left); submissions non compliant with this template will not be accepted.
- Make sure to enter your name and student I.D. number in the appropriate section of the template.
- Late submissions and submissions which do not comply with these guidelines will not be accepted.

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1. Consider a family of systems of linear equations with the *Hilbert* matrix and the dimensions $n = 2, 3, \dots, 15$. Let the right–hand side vectors in these systems have all entries equal to one. Solve these systems for all values of n using the LU decomposition algorithm

   (a) without pivoting,

   (b) with pivoting.

   Plot the 2–norm of the "defect" of the LU decomposition with and without the pivoting, respectively, $\|\mathbb{L}\mathbb{U} - \mathbb{P}\mathbb{A}\|_2$ and $\|\mathbb{L}\mathbb{U} - \mathbb{A}\|_2$ as a function of the problem size n. Next, in order to illustrate how the solutions obtained without pivoting ($\mathbf{x}_1$) and with pivoting ($\mathbf{x}_2$) diverge, plot the quantity $\frac{\|\mathbf{x}_2 - \mathbf{x}_1\|_\infty}{\|\mathbf{x}_2\|_\infty}$ as a function of the system size n. For the latter plot use the logarithmic scale for the ordinate axis. In the solution of the problem use the codes discussed at the lecture and available from the course webpage.
   [3 points]

2. Consider another family of algebraic systems

   $$\mathbb{V}\mathbf{x}_1 = \mathbf{b}_1,$$
   $$\dots$$
   $$\mathbb{V}\mathbf{x}_8 = \mathbf{b}_8$$

   now with a fixed dimension $N = 10$ and the Vandermonde matrix given by the expression

   $$[\mathbb{V}]_{i,j} = x_i^{N-j}, \qquad i, j = 1, \dots, N,$$

   where $x_i = 0.1i$. The right–hand side vectors $\mathbf{b}_k$, $k = 1, \dots, 8$ are given by the expression

   $$[\mathbf{b}_k]_i = \cos(k\pi x_i), \qquad i = 1, \dots, N, \quad k = 1, \dots, 8.$$

   Propose and implement a *computationally efficient* approach to solve this family of systems of equations. Plot all 8 solutions on a single figure. In the solution of the problem use the codes discussed at the lecture and available from the course webpage.
   [2 points]

3. You are given an initial–value problem for an ordinary differential equation

$$\frac{dy(t)}{dt} + 2y(t) = 0 \qquad \text{for } t \in (0,1],$$
$$y = 1 \qquad \text{at } t = 0.$$

Let $\mathbf{y} = [y_1, \ldots, y_N]^T$ be the vector with the values of the solution $y(t)$ at the discrete points $t_1, \ldots, t_N$, i.e., $y_i = y(t_i)$, for $i = 1, \ldots, N$. The discrete points are defined as $t_i = i\Delta t$ for $i = 1, \ldots, N$, where $\Delta t = \frac{1}{N}$. Using the following approximate expression for the derivative

$$\frac{dy(t_i)}{dt} \approx \frac{y(t_i) - y(t_i - \Delta t)}{\Delta t}$$

and assuming $N = 10$

(a) Construct and print a differentiation matrix $\mathbb{D}$ such that the differential equation is approximately satisfied at the set of points $t_1, \ldots, t_N$ and can be written in the form $\mathbb{D}\mathbf{y} = \mathbf{b}$. Note that $\mathbb{D}$ should be constructed as a *sparse* matrix.

(b) Determine the structure and print the right–hand side vector $\mathbf{b}$ corresponding to the above problem.

(c) Solve the resulting algebraic system $\mathbb{D}\mathbf{y} = \mathbf{b}$ using a method of your choice and plot the obtained approximate solution in the interval $[0,1]$ using the $\times$ symbols and a dotted line. For comparison, on the same figure plot (using the solid line) also the exact solution of the differential equation.

[5 points]