HOMEWORK #4

Due: March 25 (Tuesday) by midnight

Instructions:

- The assignment consists of *three* questions, worth 3, 4, and 3 points.
- Submit your assignment *electronically* (via Email) to the address `math2t03@math.mcmaster.ca`; hardcopy submissions will not be accepted.
- It is obligatory to use the MATLAB template file available at `http://www.math.mcmaster.ca/~bprotas/MATH2T03/template.m` (see also the link in the "Computer Programs" section of the course website on the left); submissions non compliant with this template will not be accepted.
- Make sure to enter your name and student I.D. number in the appropriate section of the template.
- Late submissions and submissions which do not comply with these guidelines will not be accepted.

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1. Implement a function

$$\texttt{diag\_dom} : \mathbb{M}^{N,N} \to \{0,1\}$$

where $\mathbb{M}^{N,N}$ is a linear space of matrices with the dimension $N \times N$. The function `diag_dom(A)` will determine whether the given matrix $\mathbb{A}$ is *diagonally–dominant* (1 returned), or not (0 returned). Write your code in such a way that the matrix $\mathbb{A}$ can be read from the keyboard (using the command `input`), and then passed to the function `diag_dom` as an argument.
[3 points]

2. Using the function `jacobi.m` (available for downloading at the course website) as a point of departure, develop a function implementing the Gauss–Seidel algorithm. Modify the code in such a way that the initial guess $\tilde{\mathbf{x}}$ can be provided as an argument to the new function, and the 2–norm of the residual for all iterations is returned in addition to `x` and `iter`. Then use this function to solve the following system of equations

$$\begin{bmatrix} 2N & 0 & 3 & 0 & & & & \dots & & 0 \\ 0 & 2N & 0 & 4 & 0 & & & & & \\ N & 0 & 2N & 0 & 5 & & & & & \\ 0 & N-1 & 0 & 2N & 0 & 6 & & & & \vdots \\ \vdots & & & \ddots & 0 & \ddots & 0 & \ddots & & \\ & & & & & 5 & 0 & 2N & 0 & N \\ & & & & & & 4 & 0 & 2N & 0 \\ 0 & & \dots & & & & 0 & 3 & 0 & 2N \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \\ \vdots \\ \\ \\ \\ y_N \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ \\ \vdots \\ \\ \\ 1 \\ -1 \end{bmatrix},$$

where $N = 50$ represents the dimension, using five different initial guesses defined as

$$\tilde{\mathbf{x}}_i = \begin{cases} 1, & \text{if } i = I_0, \\ 0, & \text{otherwise,} \end{cases}$$

where $I_0 \in \{10, 20, 30, 40, 50\}$. In the computation use the tolerance $\varepsilon = 10^{-6}$ and allow at most 100 iterations. For each of the initial guesses, plot the residual norm $\|\mathbb{A}\mathbf{x}^{(k)} - \mathbf{b}\|_2$ as a function of the iteration number k (use one figure with the logarithmic coordinates for the Y–axis and different colors for every line). When constructing the system matrix use the sparse matrix representation.
[4 points]

3. Consider the matrix

$$\mathbb{A} = \begin{bmatrix} 10 & -5 & 16 & 3 & 4 \\ 8 & 1 & 4 & 5 & 0 \\ 3 & 1 & 7 & 4 & 9 \\ -9 & 2 & -10 & -4 & -2 \\ 2 & 4 & 1 & 5 & 7 \end{bmatrix}.$$

   (a) Determine the projections onto the subspaces $\mathcal{S} = \text{null}(\mathbb{A})$ and its orthogonal complement $\mathcal{S}^{\perp}$; write out the entries of the matrices representing these projection together with the dimensions of these subspaces.

   (b) Use these projection operators to determine the projections of the vector $\mathbf{y} = [1\ 0\ 0\ 0\ 0]^T$ onto the subspaces $\mathcal{S}$ and $\mathcal{S}^{\perp}$.

   (c) Let $\{\mathbf{e}_1, \ldots, \mathbf{e}_l\}$ be the orthonormal set spanning the subspace $\mathcal{S}$ which was used to construct the projection operator $\mathbb{P}_{\mathcal{S}}$. Find (and print out) the coefficients $\{q_1, \ldots, q_l\}$ of the expansion of $(\mathbb{P}_{\mathcal{S}}\mathbf{y})$ with respect to this set, i.e., $\mathbb{P}_{\mathcal{S}}\mathbf{y} = \sum_{i=1}^{l} q_i \mathbf{e}_i$.

   [3 points]