TEST #1

10:30–11:20, February 5 (Tuesday) in T29–101

Make sure to put your name and ID number in the top–left corner of the answer sheet

No textbooks or notes allowed!

Write your answers on the reverse side.

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1. You are given a differentiable function $f : \mathbb{R} \to \mathbb{R}$. Assuming that the derivative of this function is approximated using the following finite-difference quotient for some $h \in \mathbb{R}$

$$\left.\frac{df}{dx}\right|_{x=x_0} \cong \frac{f(x_0+h) - f(x_0)}{h},$$

explain why this expression is likely to give inaccurate results for very small h when a finite–precision arithmetic is used.
[2 points]

2. You are given N vectors $\mathbf{v}_1, \dots, \mathbf{v}_N \in \mathbb{R}^N$. Propose (with justification!) an easily *computable* test to verify if these vectors are linearly independent. Apply this test to the vectors $\mathbf{v}_1 = [1\ \ -1\ \ 0]^T$, $v_2 = [2\ \ 3\ \ -1]^T$ and $\mathbf{v}_3 = [0\ \ 5\ \ -1]^T$.
[2 points]

3. State the relevant theorem and give the numbers of solutions admitted by the following systems of linear equations. Justify your answers.

   (a)
$$\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -4 \\ 4 \end{bmatrix}$$

   (b)
$$\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

   (c)
$$\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

HINT — you need not find the actual solutions!
[2 points]

4. What is the general expression for the *norm* of a linear map $A : \mathbb{R}^n \to \mathbb{R}^n$ assuming that the pre–image and image spaces are endowed with the norms $\|\cdot\|_p$ and $\|\cdot\|_q$, respectively. How can this norm be expressed in the special cases:

   (a) $p = q = 1$,

   (b) $p = q = \infty$.

[2 points]