TEST #2

10:30–11:20, March 18 (Tuesday) in T29–101

Make sure to put your name and ID number in the top–left corner of the answer sheet

No textbooks or notes allowed!

Write your answers on the reverse side. If you need more space, an extra sheet will be provided by the instructor.

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1. You are given a nonsingular matrix $\mathbb{A}$ with the dimension $n \times n$. Propose a *computationally efficient* method to compute the determinant $\det(\mathbb{A})$ of this matrix. State all the results required to derive this method and demonstrate why it is better than using the definition formula for $\det(\mathbb{A})$.
[1 point]

2. You are given the following *delay* equation

$$y(t) = \begin{cases} g(t) & 0 \le t \le 4\delta, \\ g(t) + \frac{1}{2} y(t - 4\delta) & 4\delta < t \le T, \end{cases} \tag{1}$$

where $g : [0,T] \to \mathbb{R}$ is a given function and $y : [0,T] \to \mathbb{R}$ is the unknown solution. Assuming that the interval $[0,T]$ is discretized with $N+1$ grid points t_i, such that $t_i = \delta(i-1)$, $i = 1,\dots,N+1$, where $\delta = \frac{T}{N}$, and $y_i = y(t_i)$, $g_i = g(t_i)$

   (a) write equation (1) as a system of linear equations characterizing the solution vector $\mathbf{y} = [y_1,\dots,y_{N+1}]^T$ at the discrete times $\{t_1,\dots,t_{N+1}\}$; what is the structure of the matrix of this system (how many diagonals does it have, where are they located)?

   (b) propose an efficient solution technique for a system of equations with such a structure.

   [3 points]

3. The vector $\mathbf{x} = [-1,-2]^T$ spans the subspace $\mathcal{S} = \{\lambda \mathbf{x},\ \lambda \in \mathbb{R}\}$ in $\mathbb{R}^2$.

   (a) Compute the matrices $\mathbb{P}_{\mathcal{S}}$ and $\mathbb{P}_{\mathcal{S}^\perp}$ representing projections on the subspaces $\mathcal{S}$ and $\mathcal{S}^\perp$.

   (b) Find the vector $\mathbf{y} \in \mathbb{R}^2$ simultaneously orthogonal to $\mathcal{S}$ and $\mathcal{S}^\perp$.

   (c) Determine the set $\operatorname{range}(\mathbb{P}_{\mathcal{S}}^*)^\perp$, where $\mathbb{P}_{\mathcal{S}}^*$ is the adjoint of the projection $\mathbb{P}_{\mathcal{S}}$.

   [3 points]

4. You are given a linearly independent set of vectors $\mathbf{y}_1,\dots\mathbf{y}_M \in \mathbb{R}^N$, $M \le N$. What computational algorithm would you use in order to determine an orthogonal set $\mathbf{x}_1,\dots\mathbf{x}_M \in \mathbb{R}^N$, such that $\operatorname{span}(\mathbf{x}_1,\dots\mathbf{x}_M) = \operatorname{span}(\mathbf{y}_1,\dots\mathbf{y}_M)$? Characterize the objects returned by this algorithm and required by it as arguments.
[1 point]