Things You Should Know (But Likely Forgot)

Trigonometric identities:

$\cos^2\theta + \sin^2\theta = 1 \qquad \tan^2\theta = \sec^2\theta - 1$

$\cos(2\,\theta) = \cos^2\theta - \sin^2\theta = 2\cos^2\theta - 1 = 1 - 2\sin^2\theta$

$\sin(2\theta) = 2\cos\theta\sin\theta,$

$\cos^2\theta = \frac{1}{2}(1+\cos(2\,\theta)), \qquad \sin^2\theta = \frac{1}{2}(1-\cos(2\,\theta))$

$\cos(a \pm b) = \cos(a)\cos(b) \mp \sin(a)\sin(b)$

$\sin(a \pm b) = \sin(a)\cos(b) \pm \cos(a)\sin(b)$

$\sin(a)\cos(b) = \frac{1}{2}(\sin(a+b) + \sin(a-b))$

$\cos(a)\sin(b) = \frac{1}{2}(\sin(a+b) - \sin(a-b))$

$\cos(a)\cos(b) = \frac{1}{2}(\cos(a+b) + \cos(a-b))$

$\sin(a)\sin(b) = \frac{1}{2}(\cos(a-b) - \cos(a+b))$

$\sin(a) + \sin(b) = 2\sin\left(\frac{a+b}{2}\right)\cos\left(\frac{a-b}{2}\right)$

$\sin(a) - \sin(b) = 2\cos\left(\frac{a+b}{2}\right)\sin\left(\frac{a-b}{2}\right)$

$\cos(a) + \cos(b) = 2\cos\left(\frac{a+b}{2}\right)\cos\left(\frac{a-b}{2}\right)$

$\cos(a) - \cos(b) = -2\sin\left(\frac{a+b}{2}\right)\sin\left(\frac{a-b}{2}\right)$

$\cos\theta = \frac{e^{i\theta}+e^{-i\theta}}{2} \qquad \sin\theta = \frac{e^{i\theta}-e^{-i\theta}}{2i} \; e^{\pm i\theta} = \cos\theta \pm i\sin\theta$

$(\cos\theta)' = -\sin\theta \qquad (\sin\theta)' = \cos\theta$

$(\tan\theta)' = \sec^2\theta \qquad (\sec\theta)' = \sec\theta\tan\theta$

Weierstrass substitution:

$t = \tan\left(\frac{x}{2}\right)$ makes $dx = \frac{2}{1+t^2}dt, \qquad \cos x = \frac{1-t^2}{1+t^2}, \qquad \sin x = \frac{2t}{1+t^2}$

Hyperbolics:

$e^{\pm x} = \cosh x \pm \sinh x \qquad \begin{cases} \cos x \text{ even}, & \cosh x \text{ even} \\ \sin x \text{ odd}, & \sinh x \text{ odd} \end{cases}$

$\cosh x = \frac{e^x+e^{-x}}{2}, \qquad \sinh x = \frac{e^x-e^{-x}}{2}, \qquad \tanh x = \frac{\sinh x}{\cosh x}, \qquad \operatorname{sech} x = \frac{1}{\cosh x}$

$\cosh^2 x - \sinh^2 x = 1, \qquad \tanh^2 x = 1 - \operatorname{sech}^2 x$

"Arctrig" (inverse trigonometric functions):

$\sin^{-1}(x)$ has domain $[-1, 1]$, range $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ $\quad (\sin^{-1}(x))' = \frac{1}{\sqrt{1-x^2}}$

$\cos^{-1}(x)$ has domain $[-1, 1]$, range $[0, \pi]$ $\quad (\cos^{-1}(x))' = \frac{-1}{\sqrt{1-x^2}}$

$\tan^{-1}(x)$ has domain $(-\infty, \infty)$, range $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ $\quad (\tan^{-1}(x))' = \frac{1}{1+x^2}$

Inverse hyperbolics:

$\sinh^{-1}(x) = \ln\left(x + \sqrt{x^2+1}\right)$, domain and range $(-\infty, \infty)$, $\quad (\sinh^{-1}(x))' = \frac{1}{\sqrt{x^2+1}}$

$\cosh^{-1}(x) = \ln\left(x + \sqrt{x^2-1}\right)$, domain $[1, \infty)$, range $[0, \infty)$ $\quad (\cosh^{-1}(x))' = \frac{1}{\sqrt{x^2-1}}$

$\tanh^{-1}(x) = \frac{1}{2}\ln\frac{1+x}{1-x}$, domain $(-1, 1)$, range $(-\infty, \infty)$ $\quad (\tanh^{-1}(x))' = \frac{1}{1-x^2}$

Useful integration facts and tricks:

$\int f(x)\,dx = \int_{\xi}^{x} f(t)\,dt + C$ for a ξ of your choice (at which the expression is well-defined)

$\int_a^b u\,dv = (uv)|_a^b - \int_a^b v\,du$ ...integration by parts

$\int xe^x dx = (x-1)e^x + C$ $\qquad \int \ln x\,dx = x(\ln x - 1) + C$

$\int \frac{1}{a-bx} dx = -\frac{1}{b}\ln|a-bx| + C$

$\int \frac{1}{a^2+x^2} dx = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right) + C$ $\qquad \int \frac{1}{\sqrt{a^2-x^2}} dx = \sin^{-1}\left(\frac{x}{a}\right) + C$

$\int \frac{1}{a^2-x^2} dx = \frac{1}{2a}\ln\left|\frac{a-x}{a+x}\right| + C$

$\int \frac{1}{\sqrt{a^2+x^2}} dx = \sinh^{-1}\left(\frac{x}{a}\right) + C$ $\qquad \int \frac{1}{\sqrt{x^2-a^2}} dx = \cosh^{-1}\left(\frac{x}{a}\right) + C$

One variable at a time:

if $f(x,y) = \frac{\partial}{\partial y}F(x,y)$ then $\int f(x,y)dy = F(x,y) + g(x)$

where g is an arbitrary function of x.

It works with more variables: $\int \frac{\partial F(x,y,z)}{\partial y} dy = F(x,y,z) + g(x,z)$.

Determinant tricks: for $n \times n$ matrices,

$$\det\begin{bmatrix} \text{row}_1 \\ \vdots \\ t\,\text{row}_j \\ \vdots \\ \text{row}_n \end{bmatrix} = t\det\begin{bmatrix} \text{row}_1 \\ \vdots \\ \text{row}_j \\ \vdots \\ \text{row}_n \end{bmatrix}$$

(multiplying a row by t multiplies the determinant by t)

$$\det\begin{bmatrix} \text{row}_1 \\ \vdots \\ \text{row}_i \\ \vdots \\ \text{row}_j \\ \vdots \\ \text{row}_n \end{bmatrix} = (-1)\det\begin{bmatrix} \text{row}_1 \\ \vdots \\ \text{row}_j \\ \vdots \\ \text{row}_i \\ \vdots \\ \text{row}_n \end{bmatrix}$$

(swapping any two rows flips the sign of the determimamt)

$$\det\begin{bmatrix} \text{row}_1 \\ \vdots \\ \text{row}_i \\ \vdots \\ \text{row}_j + t\,\text{row}_i \\ \vdots \\ \text{row}_n \end{bmatrix} = \det\begin{bmatrix} \text{row}_1 \\ \vdots \\ \text{row}_i \\ \vdots \\ \text{row}_j \\ \vdots \\ \text{row}_n \end{bmatrix}$$

(adding a multiple of one row to another row does not change the determinant)

Shortcut:

$$\det\begin{bmatrix} a & * & * & * \\ 0 & b & c & \cdots \\ 0 & d & \cdots & \cdots \\ \vdots & \cdots & \cdots & \cdots \end{bmatrix} = a\det\begin{bmatrix} b & c & \cdots \\ d & \cdots & \cdots \\ \cdots & \cdots & \cdots \end{bmatrix}$$

where $*$ means "any number".

Also: $\det(tA) = t^n\det(A)$ (where A is an $n \times n$ matrix)

$\det(A^T) = \det(A)$ (transpose property):

in A^T, elements get flipped across the diagonal, like $\begin{bmatrix} a & b \\ c & d \end{bmatrix}^T = \begin{bmatrix} a & c \\ b & d \end{bmatrix}$

Shortcut for inverting a 2×2 matrix:

if $ad - bc \neq 0$, $\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$.

Linear dependence, Spanning and Algebra Issues

w is in the span of u and v iff for some $a,\, b \in \mathbb{R}$,

$$w = au + bv$$

The set of all such w is the span of u and v.

Extended: The span of $\{x_1, ..., x_n\}$ is $\left\{ w \,|\, w = \sum_{i=1}^{n} a_i x_i \;,\; a_i \in \mathbb{R} \right\}$

A set (list) $\{x_1, ..., x_n\}$ of vectors is "LI" (linearly independent) if the equality $\sum_{i=1}^{n} a_i x = 0$ is satisfied only by making all $a_i = 0$.

Note that no vector in a LI set is a linear combination of the others (i.e. not in the span of the others).

And specifically, no vector in a LI set can be a multiple of any of the others, either.

For 2 vectors: LI simply means that $x_1 \neq$ const. $\cdot\, x_2$ and $x_2 \neq$ const. $\cdot\, x_1$ is sufficient to demonstrate that $\{x_1\,,\, x_2\}$ is LI.

Warning: For 3 or more vectors, one must be more subtle: for example

$\{x^2,\; 1 - 3x^2,\; 1\}$ is not LI, but none is a multiple of any other, either.

A basis of a span of vectors is the smallest (by number of members) set of vectors that can be used to span the same space. Note that a basis is not unique.

For example, $\left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}$ spans $\mathbb{R}^2$, as does $\left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \end{bmatrix} \right\}$, etc.

These bases are inevitably LI, and all have the same number of vectors; this number is the dimension of the span. For example, $\mathbb{R}^2$ has dimension 2.

$\rightarrow$ Any LI set of vectors in a space, with the same number of vectors as the dimension is, in fact, a basis.

A time-saving trick:

Linear independence of "sub-vectors" implies LI of the larger vectors, but linear dependence of sub-vectors proves nothing.

Example:

$$\text{let } \vec{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 1 \end{bmatrix}, \quad \vec{v}_2 = \begin{bmatrix} 0 \\ 1 \\ 2 \\ 2 \end{bmatrix}, \quad \vec{u}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \; \vec{u}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

The elements of $\vec{u}_1$ are the first two elements of $\vec{v}_1$, similarly for $\vec{u}_2$ and $\vec{v}_2$.

Clearly $\vec{u}_1$, $\vec{u}_2$ are not proportional, so $\vec{v}_1$, $\vec{v}_2$ are not proportional, without having to check the rest of the elements.

But: the converse is no good: the vectors from the last two elements of $\vec{v}_1$, $\vec{v}_2$ are proportional, but $\vec{v}_1$ and $\vec{v}_2$ are LI.

Linear operators

$f(x)$ is a linear function (or operator) iff

$f(x_1+x_2)=f(x_1)+f(x_2)$ x_1, x_2 are vectors, or functions

$f(kx)=kf(x)$ $k\in\mathbb{R}$

(Note: This implies $f(0)=0$.)

Condensed form: $f(k\vec{x}_1+l\vec{x}_2)=kf(\vec{x}_1)+lf(\vec{x}_2)$

The only type of example in $\mathbb{R}$: $f(x)=mx,\quad m\in\mathbb{R}$.

In $\mathbb{R}^n$, every matrix A with n columns represents a linear operator:

$A(\vec{x}+\vec{y})=A\vec{x}+A\vec{y}$, and $A(k\vec{x})=kA\vec{x}$.

Note: If f and g are linear then so are $f+g$, $kf\,(k\in\mathbb{R})$ and so is the composition $f\circ g$ where $(f\circ g)(\vec{x})=f(g(\vec{x}))$.

For example: $\frac{d}{dx}$ is a linear operator on functions, as is $\left(3\frac{d^2}{dx^2}+\frac{d}{dx}\right)$.

Polar form

$(x,\,y)\to(r,\theta)$ (cartesian to polar) Usually we assume $r>0$, $\theta\in[0,2\pi)$ or $(-\pi,\pi]$ to maintain uniqueness for all points other than $(0,\,0)$ (but sometimes it is useful to bend the rule)

$r^2=x^2+y^2$, so $r=\sqrt{x^2+y^2}$, $\cos\theta=x/r,\quad\sin\theta=y/r$,

if $\theta\in\left(-\frac{\pi}{2},\,\frac{\pi}{2}\right)$, that is, $x>0$, then $\theta=\tan^{-1}\left(\frac{y}{x}\right)$ (note the restriction!!!)

$x=r\cos\theta,\quad y=r\sin\theta$ (the underlined formulas are by far the most useful ones).

Vectors and geometry

For $\vec{u}$, $\vec{v}\in\mathbb{R}^3$, their cross-product is $\vec{u}\times\vec{v}=\begin{vmatrix}\vec{i} & \vec{j} & \vec{k}\\ u_1 & u_2 & u_3\\ v_1 & v_2 & v_3\end{vmatrix}$

Properties

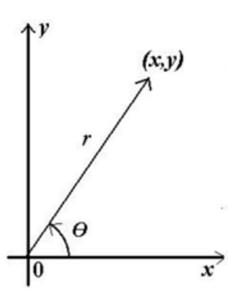
$\|\vec{u}\times\vec{v}\|=\|\vec{u}\|\|\vec{v}\|\sin\theta$ (θ being the angle from $\vec{u}$ to $\vec{v}$)

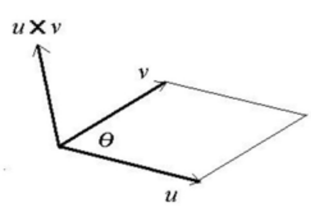
$\vec{u}\perp(\vec{u}\times\vec{v})$ and $\vec{v}\perp(\vec{u}\times\vec{v})$, the right-hand rule determines the direction.

$\vec{u}\times\vec{v}=-\vec{v}\times\vec{u}$,

$(k\vec{u})\times\vec{v}=k(\vec{u}\times\vec{v})$, $\vec{u}\times(\vec{v}+\vec{w})=(\vec{u}\times\vec{v})+(\vec{u}\times\vec{w})$

Area of a parallelogram with sides $\vec{u}$, $\vec{v}$ is $\|\vec{u}\times\vec{v}\|=\|\vec{u}\|\|\vec{v}\|\sin\theta$.





Dot product:

If $\vec{u},\, \vec{v} \in \mathbb{R}^n$ then $\vec{u}\cdot\vec{v} = \sum_{i-1}^{n} u_i v_i$ (sum of products of respective coordinates)

Properties:

$$\vec{u}\cdot\vec{v} = \vec{v}\cdot\vec{u},\;\; (k\vec{u})\cdot\vec{v} = k(\vec{u}\cdot\vec{v}),\;\; \vec{u}\cdot(\vec{v}+\vec{w}) = \vec{u}\cdot\vec{v}+\vec{u}\cdot\vec{w}$$

$$\vec{u}\cdot\vec{v} = \|\vec{u}\|\|\vec{v}\|\cos\theta,\; \|\vec{u}\| = \sqrt{\vec{u}\cdot\vec{u}} = \sqrt{\sum_{i=1}^{n} u_i^2}$$

Volume of a parallelepiped with edges $\vec{u},\, \vec{v},\, \vec{w}$: $|\vec{u}\cdot(\vec{v}\times\vec{w})|$

Projection: $\text{proj}_{\vec{v}}\vec{u} = \frac{\vec{u}\cdot\vec{v}}{\vec{v}\cdot\vec{v}}\vec{v}$, component $\text{comp}_{\vec{v}}\vec{u} = \|\vec{u}\|\cos\theta = \frac{\vec{u}\cdot\vec{v}}{\|\vec{v}\|}$

Inequalities:

$$\left|\vec{a}\cdot\vec{b}\right| \le \|\vec{a}\|\left\|\vec{b}\right\| \quad \text{(Cauchy-Schwarz Inequality)}$$

$$\left\|\vec{a}+\vec{b}\right\| \le \|\vec{a}\| + \left\|\vec{b}\right\| \quad \text{(Triangle Inequality)}$$

Partial Fractions: For example,

$$\frac{x^3-5x^2+2x}{x(x-1)^3(x^2+1)^2} = \frac{A}{x} + \frac{B}{x-1} + \frac{C}{(x-1)^2} + \frac{D}{(x-1)^3} + \frac{Ex+F}{x^2+1} + \frac{Gx+H}{(x^2+1)^2}$$

Note: (1) Degree of numerator < degree of denominator, i.e., $3 < 1+3+2\times 2$

(2) The powers of factors go from 1 to the original power

To find $A,\, B,\, C,\, D.\, E.\, F,\, G,\, H$, clear the fractions, expand the powers and compare coefficients.

You obtain a system of linear equations, same number of equations as unknowns, and non-singular (I promise).

On an example: $\frac{x+6}{(x+1)^2(x+3)} = \frac{A}{x+1} + \frac{B}{(x+1)^2} + \frac{C}{x+3}$

$$x+6 = A(x+1)(x+3) + B(x+3) + C(x+1)^2 \qquad \text{line 1}$$
$$= A(x^2+4x+3) + B(x+3) + C(x^2+2x+1) \qquad \text{line 2}$$
$$x+6 = (A+C)x^2 + (4A+B+2C)x + (3A+3B+C) \qquad \text{line 3}$$

System: $A+C=0,\;\; 4A+B+2C=1,\;\; 3A+3B+C=6$

Solution (any correct method): $A = -\frac{3}{4}\,,\;\; B = \frac{5}{2}\,,\;\; C = \frac{3}{4}$

Result:

$$\frac{x+6}{(x+1)^2(x+3)} = -\frac{3}{4}\cdot\frac{1}{x+1} + \frac{5}{2}\cdot\frac{1}{(x+1)^2} + \frac{3}{4}\cdot\frac{1}{x+3}$$

Some shortcuts:

In the factored form (line 1), set $x = -1$ to obtain quickly $5 = 2B$, and so on.

You may not get all constants this way, but the ones you do get can speed up the calculations, say by replacing the unknowns in line 3 by their values.

Taylor Series: $f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n$ (where f is "analytic" and the series is convergent), When $a=0$, the series is also caled Maclaurin series.

Handy short forms:

&	"ampersand", meaning "and"
const.	constant
cont., cts.	continuous (cont. for "continuity")
DE	differential equation
dec., decr.	decreasing
diff.	differentiate, differentiable
dim	dimension
equ.	equation, equal
fcn, f'n	function
FTC	Fundamental Theorem of Calculus
fund.	fundamental
iff	if and only if
inc., incr.	increasing
LI	linearly independent (or linear independence)
MVT	Mean Value Theorem
ODE	ordinary differential equation
PDE	partial differential equation
pf.	proof
situ.	situation
soln.	solution
Thm	theorem
var, Var	variance, variable