Solutions to some of the problems in Chapters 15 and 16 of Cain & Herod.

1 Section 15.1

15.1, Problem 1

We use $x = u$ and $y = v$ as parameters. The surface can then be parametrized by

$$\mathbf{r}(u,v) = \langle u, v, \sqrt{u + 2\,v^2} \rangle, \quad u, v \geq 0.$$

15.1, Problem 2

If we let $x' = 2\,x$, $y' = y$ and $z' = 2\sqrt{2}\,z$, the equation becomes $(x')^2 + (y')^2 + (z')^2 = 4^2$, which is a sphere centered at the origin with radius 4 in the x', y', z' coordinates. We can use spherical coordinates to parametrize the sphere:

$$x' = 4\,\cos\theta\,\sin\phi, \quad y' = 4\,\sin\theta\,\sin\phi, \quad z' = 4\,\cos\phi, \quad 0 \leq \theta \leq 2\,\pi,\; 0 \leq \phi \leq \pi.$$

Reverting back to the x, y, z coordinates, we have thus

$$x = 2\,\cos\theta\,\sin\phi, \quad y = 4\,\sin\theta\,\sin\phi, \quad z = \sqrt{2}\,\cos\phi, \quad 0 \leq \theta < 2\,\pi,\; 0 \leq \phi \leq \pi.$$

or, letting $u = \theta$ and $v = \phi$,

$$\mathbf{r}(u,v) = \langle 2\,\cos u\,\sin v,\, 4\,\sin u\,\sin v,\, \sqrt{2}\,\cos v \rangle, \quad 0 \leq u \leq 2\,\pi,\; 0 \leq v \leq \pi.$$

15.1, Problem 3

If $x^2 + y^2 = 1$, we can write, using polar coordinates that $x = \cos\theta$ and $y = \sin\theta$ for some θ with $0 \leq \theta < 2\,\pi$. Using $u = \theta$ and $v = z$ as parameters, the cylinder can thus be parametrized by

$$\mathbf{r}(u,v) = \langle \cos u,\, \sin u,\, v \rangle, \quad 0 \leq u \leq 2\,\pi,\; -\infty < v < \infty.$$

15.1, Problem 4

If the surface is parametrized by

$$\mathbf{r}(u,v) = \langle u\,\cos v,\, u\,\sin v,\, u \rangle, \quad 0 \leq v \leq 2\,\pi,\; -1 \leq u \leq 1,$$

we have $x^2 + y^2 = u^2\cos^2 v + u^2\sin^2 v = u^2 = z^2$. The surface is thus the part of the hyperboloid $x^2 + y^2 = z^2$ between the planes $z = -1$ and $z = 1$.

15.1, Problem 5

If the surface is parametrized by

$$\mathbf{r}(u, v) = \langle u\cos v,\ u\sin v,\ u^2\rangle, \quad 0 \le v \le 2\pi,\ 1 \le u \le 2,$$

we have $x^2 + y^2 = u^2\cos^2 v + u^2\sin^2 v = u^2 = z$. The surface is thus the part of the paraboloid $z = x^2 + y^2$ between the planes $z = 1$ and $z = 2$.

15.1, Problem 6

The equation for the sphere of radius 3 centered at the point (1, 2, 3) is

$$(x-1)^2 + (y-2)^2 + (z-3)^2 = 3^2.$$

If we let $x' = x - 1$, $y' = y - 2$ and $z' = z - 3$, we get the equation $(x')^2 + (y')^2 + (z')^2 = 3^2$, which is a sphere centered at the origin with radius 3 in the x', y', z' coordinates. We can use spherical coordinates to parametrize the sphere:

$$x' = 3\cos\theta\sin\phi, \quad y' = 3\sin\theta\sin\phi, \quad z' = 3\cos\phi, \quad 0 \le \theta \le 2\pi,\ 0 \le \phi \le \pi.$$

Reverting back to the x, y, z coordinates, we have thus

$$x = 1 + 3\cos\theta\sin\phi, \quad y' = 2 + 3\sin\theta\sin\phi, \quad z' = 3 + 3\cos\phi, \quad 0 \le \theta \le 2\pi,\ 0 \le \phi \le \pi.$$

or, letting $u = \theta$ and $v = \phi$,

$$\mathbf{r}(u, v) = \langle 1 + 3\cos u\sin v,\ 2 + 3\sin u\sin v,\ 3 + 3\cos v\rangle, \quad 0 \le u \le 2\pi,\ 0 \le v \le \pi.$$

15.1, Problem 7

We use the parametrization

$$x = a\cos\theta\sin\phi, \quad y = a\sin\theta\sin\phi, \quad z = a\cos\phi, \quad 0 \le \theta \le 2\pi,\ 0 \le \phi \le \pi.$$

or

$$\mathbf{r}(\phi, \theta) = \langle a\cos\theta\sin\phi,\ a\sin\theta\sin\phi,\ a\cos\phi\rangle, \quad 0 \le \theta \le 2\pi,\ 0 \le \phi \le \pi.$$

We have

$$\mathbf{r}_\phi = \langle a\cos\theta\cos\phi,\ a\sin\theta\cos\phi,\ -a\sin\phi\rangle,$$

and

$$\mathbf{r}_\theta = \langle -a\sin\theta\sin\phi,\ a\cos\theta\sin\phi, 0\rangle.$$

Thus,

$$\mathbf{r}_\phi \times \mathbf{r}_\theta = \langle a^2\cos\theta\sin^2\phi,\ a^2\sin\theta\sin^2\phi,\ a^2\cos\phi\sin\phi\rangle = a\sin\phi\,\mathbf{r}(\phi, \theta).$$

This implies that the normal vector $\mathbf{r}_\phi \times \mathbf{r}_\theta$ to the sphere at a point P on the surface has the same direction has the direction vector of that point, $\overrightarrow{OP}$. In particular, if $P = (\frac{a}{\sqrt{3}}, \frac{a}{\sqrt{3}}, -\frac{a}{\sqrt{3}})$, the normal vector will have the same direction as $\langle \frac{a}{\sqrt{3}}, \frac{a}{\sqrt{3}}, -\frac{a}{\sqrt{3}} \rangle$ or as $\langle 1, 1, -1 \rangle$. The normal line at P as thus vector equation

$$\mathbf{s}(t) = \langle \frac{a}{\sqrt{3}}, \frac{a}{\sqrt{3}}, -\frac{a}{\sqrt{3}} \rangle + t \langle 1, 1, -1 \rangle, \quad -\infty < t < \infty,$$

or, more simply,

$$\mathbf{s}(t) = t \langle 1, 1, -1 \rangle, \quad -\infty < t < \infty.$$

15.1, Problem 8

Following problem 7 above, we have that the vector $\langle 1, 1, -1 \rangle$ is normal to the sphere at the point $P = (\frac{a}{\sqrt{3}}, \frac{a}{\sqrt{3}}, -\frac{a}{\sqrt{3}})$. The tangent plane at P has thus equation

$$(x - \frac{a}{\sqrt{3}}) + (y - \frac{a}{\sqrt{3}}) - (z + \frac{a}{\sqrt{3}}) = 0 \quad \text{or} \quad x + y - z = \sqrt{3}\, a.$$

15.1, Problem 9

To find the points on the surface parametrized by

$$\mathbf{r}(s, t) = (s^2 + t^2)\, \mathbf{i} + (s + 3\, t)\, \mathbf{j} - s\, t\, \mathbf{k}$$

at which the tangent plane is parallel to the plane $5\, x - 6\, y + 2\, z = 7$, we fist compute the normal vector $\mathbf{r}_s \times \mathbf{r}_t$. We have

$$\mathbf{r}_s = (2\, s)\, \mathbf{i} + \mathbf{j} - t\, \mathbf{k}$$

and

$$\mathbf{r}_t = (2\, t)\, \mathbf{i} + 3\, \mathbf{j} - s\, \mathbf{k}.$$

Thus,

$$\mathbf{r}_s \times \mathbf{r}_t = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2\, s & 1 & -t \\ 2\, t & 3 & -s \end{vmatrix} = (3\, t - s)\, \mathbf{i} + 2\, (s^2 - t^2)\, \mathbf{j} + (6\, s - 2\, t)\, \mathbf{k}$$

For the planes to be parallel, we need their normal vectors to be parallel. There must thus exist a constant $\lambda \neq 0$, such that

$$(3\, t - s)\, \mathbf{i} + 2\, (s^2 - t^2)\, \mathbf{j} + (6\, s - 2\, t)\, \mathbf{k} = \lambda\, (5\, \mathbf{i} - 6\, \mathbf{j} + 2\, \mathbf{k}).$$

or

$$3\, t - s = 5\, \lambda, \quad 2\, (s^2 - t^2) = -6\, \lambda, \quad 6\, s - 2\, t = 2\, \lambda.$$

Solving first the 1st and 3rd equation yields $t = 2\, \lambda$ and $s = \lambda$. Replacing s, t in terms of λ in the 2d equation yields $-6\, \lambda^2 = -6\lambda$ or $\lambda = 0$ or 1. Since $\lambda \neq 0$, we have thus $\lambda = 1$

which implies that $s = 1$ and $t = 2$. the point on the surface corresponding to the parameter values is thus $P = (5,\ 7,\ -2)$.

15.1, Problem 10

If the surface is parametrized by

$$\mathbf{r}(s,t) = (s+t)\,\mathbf{i} + (s^2)\,\mathbf{j} - 2\,t^2\,\mathbf{k},$$

we have

$$\mathbf{r}_s = \mathbf{i} + (2\,s)\,\mathbf{j}$$

and

$$\mathbf{r}_t = \mathbf{i} - 4\,t\,\mathbf{k}.$$

Thus,

$$\mathbf{r}_s \times \mathbf{r}_t = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 2\,s & 0 \\ 1 & 0 & -4\,t \end{vmatrix} = (-8\,s\,t)\,\mathbf{i} + (4\,t)\,\mathbf{j} + (-2\,s)\,\mathbf{k}$$

The point $P = (1,\ 4,\ -18)$ on the surface corresponds to parameters s, t such that

$$s + t = 1, \quad s^2 = 4, \quad -2\,t^2 = -18.$$

From the last two equation, we get $s = \pm 2$ and $t = \pm 3$ and for the 1st equation to be satisfies, the only possibility is take $t = 3$ and $s = -2$. A vector normal to the surface at the point $P = (1,\ 4,\ -18)$ is thus

$$(\mathbf{r}_s \times \mathbf{r}_t)\,(-2, 3) = \langle 48,\ 12,\ 4\rangle$$

or, more simply, the vector $\langle 12,\ 3,\ 1\rangle$. The equation for the plane parallel to that tangent plane to the surface at $(1,\ 4,\ -18)$ and containing the point $(1,\ -2,\ 3)$ is thus

$$12\,(x-1) + 3\,(y+2) + (z-3) = 0 \quad \text{or} \quad 12\,x + 3\,y + z = 9.$$

2 Section 15.2

15.2, Problem 1

We parametrize S by

$$\mathbf{r}(x,y) = \langle x,\ y,\ x^2+y^2\rangle, \quad 1 \le x^2+y^2 \le 2.$$

We have

$$\mathbf{r}_x = \langle 1,\ 0,\ 2\,x\rangle \quad \text{and} \quad \mathbf{r}_y = \langle 0,\ 1,\ 2\,y\rangle.$$

Hence,

$$\mathbf{r}_x \times \mathbf{r}_y = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 0 & 2\,x \\ 0 & 1 & 2\,y \end{vmatrix} = (-2\,x)\,\mathbf{i} + (-2\,y)\,\mathbf{j} + \mathbf{k}$$

and $\|\mathbf{r}_x \times \mathbf{r}_y\| = \sqrt{4\,x^2 + 4\,y^2 + 1}$. Letting $D = \{(x,y),\ 1 \leq x^2 + y^2 \leq 2\}$, we have thus

$$A(S) = \int_S 1\,dS = \iint_D \|\mathbf{r}_x \times \mathbf{r}_y\|\,dA = \iint_D \sqrt{4\,(x^2+y^2)+1}\,dA$$

Passing to polar coordinates, we obtain

$$\begin{aligned} A(S) &= \int_0^{2\,\pi} \int_1^{\sqrt{2}} \sqrt{4\,r^2+1}\,r\,dr\,d\theta = 2\,\pi\,\left[\frac{1}{12}(4\,r^2+1)^{3/2}\right]_{r=1}^{r=\sqrt{2}} \\ &= \frac{\pi}{6}\,(9^{3/2} - 5^{3/2}) = \frac{\pi}{6}\,(27 - 5\,\sqrt{5}) \end{aligned}$$

15.2, Problem 1

The centroid is the point $(\overline{x}, \overline{y}, \overline{z})$, where

$$\overline{x} = \frac{\int_S x\,dS}{A(S)}, \quad \overline{y} = \frac{\int_S y\,dS}{A(S)}, \quad \overline{z} = \frac{\int_S z\,dS}{A(S)}.$$

We have

$$\int_S x\,dS = \iint_D x\,\|\mathbf{r}_x \times \mathbf{r}_y\|\,dA = \iint_D x\,\sqrt{4\,(x^2+y^2)+1}\,dA$$

Passing to polar coordinates, we obtain

$$\begin{aligned} \int_S x\,dS &= \int_0^{2\,\pi} \int_1^{\sqrt{2}} r\,\cos\theta\,\sqrt{4\,r^2+1}\,r\,dr\,d\theta = \int_0^{2\,\pi} \cos\theta\,d\theta \int_1^{\sqrt{2}} r^2\,\sqrt{4\,r^2+1}\,dr \\ &= [\sin\theta]_0^{2\,\pi} \int_1^{\sqrt{2}} r^2\,\sqrt{4\,r^2+1}\,dr = 0 \end{aligned}$$

since the first factor in the product is zero. Similarly,

$$\int_S y\,dS = \iint_D x\,\|\mathbf{r}_y \times \mathbf{r}_y\|\,dA = \iint_D y\,\sqrt{4\,(x^2+y^2)+1}\,dA$$

and, after passing to polar coordinates,

$$\begin{aligned} \int_S y\,dS &= \int_0^{2\,\pi} \int_1^{\sqrt{2}} r\,\sin\theta\,\sqrt{4\,r^2+1}\,r\,dr\,d\theta = \int_0^{2\,\pi} \sin\theta\,d\theta \int_1^{\sqrt{2}} r^2\,\sqrt{4\,r^2+1}\,dr \\ &= [-\cos\theta]_0^{2\,\pi} \int_1^{\sqrt{2}} r^2\,\sqrt{4\,r^2+1}\,dr = 0. \end{aligned}$$

Finally,

$$\int_S z\,dS = \iint_D (x^2+y^2)\,\|\mathbf{r}_y \times \mathbf{r}_y\|\,dA = \iint_D (x^2+y^2)\,\sqrt{4\,(x^2+y^2)+1}\,dA$$

and, after passing to polar coordinates,

$$\begin{aligned}
\int_S z\,dS &= \int_0^{2\pi}\int_1^{\sqrt{2}} r^2\,\sqrt{4\,r^2+1}\,r\,dr\,d\theta = 2\,\pi \int_1^{\sqrt{2}} r^3\,\sqrt{4\,r^2+1}\,dr \\
&= 2\,\pi \int_1^{\sqrt{2}} r\,\left(\frac{4r^2+1}{4} - \frac{1}{4}\right)\,\sqrt{4\,r^2+1}\,dr \\
&= 2\,\pi \int_1^{\sqrt{2}} r\,\left(\frac{(4r^2+1)^{3/2}}{4} - \frac{(4r^2+1)^{1/2}}{4}\right)\,dr \\
&= 2\,\pi\,\left[\frac{(4r^2+1)^{5/2}}{80} - \frac{(4r^2+1)^{3/2}}{48}\right]_1^{\sqrt{2}} \\
&= 2\,\pi\,\left(\frac{9^{5/2}-5^{5/2}}{80} - \frac{9^{3/2}-5^{3/2}}{48}\right) \\
&= 2\,\pi\,\left(\frac{243}{80} - \frac{27}{48} + \frac{5\sqrt{5}}{48} - \frac{25\sqrt{5}}{80}\right) \\
&= \pi\,\left(\frac{99}{20} - \frac{5}{12}\sqrt{5}\right)
\end{aligned}$$

Therefore,

$$\overline{z} = \frac{\int_S z\,dS}{A(S)} = \frac{\pi\,\left(\frac{99}{20} - \frac{5}{12}\sqrt{5}\right)}{\frac{\pi}{6}\,(27-5\sqrt{5})} = 6\,\frac{\left(\frac{99}{20} - \frac{5}{12}\sqrt{5}\right)(27+5\sqrt{5})}{604}$$

and the centroid is

$$(\overline{x},\,\overline{y},\,\overline{z}) = \left(0,\,0,\,6\,\frac{\left(\frac{99}{20} - \frac{5}{12}\sqrt{5}\right)(27+5\sqrt{5})}{604}\right).$$

15.2, Problem 16

The surface can be parametrized by

$$\mathbf{r}(x,y) = \langle x,\,y,\,2\,x\rangle, \quad \text{for } x^2+y^2 \le 1.$$

We have

$$\mathbf{r}_x = \langle 1,\,0,\,2\rangle \quad \text{and} \quad \mathbf{r}_y = \langle 0,\,1,\,0\rangle$$

Hence,

$$\mathbf{r}_x \times \mathbf{r}_y = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 0 & 2 \\ 0 & 1 & 0 \end{vmatrix} = (-2)\,\mathbf{i} + \mathbf{k} \quad \text{and} \quad \|\mathbf{r}_x \times \mathbf{r}_y\| = \sqrt{5}.$$

Therefore, letting $D = \{(x, y),\ x^2 + y^2 \le 1\}$, we have

$$A(S) = \iint_S 1\, dS = \iint_D \|\mathbf{r}_x \times \mathbf{r}_y\|\, dA = \int_D \sqrt{5}\, dA = \int_0^{2\pi} \int_0^1 \sqrt{5}\, r\, dr\, d\theta = \pi\sqrt{5}.$$

15.2, Problem 18

The integral to be computed should be replaced by

$$\iint_S x\sqrt{4 + y^2}\, dS$$

(Otherwise, it can't be computed explicitely) Note that the paraboloid $y^2+4\,z = 16$ intersects the plane $z = 0$ when $y = \pm 4$. The surface is thus the graph of the function $z = 4 - y^2/4$ defined for $0 \le x \le 1$ and $-4 \le y \le 4$. It is thus parametrized by

$$\mathbf{r}(x, y) = \langle x,\ y,\ 4 - y^2/4\rangle, \quad \text{for } 0 \le x \le 1,\ -4 \le y \le 4.$$

$$\mathbf{r}_x = \langle 1,\ 0,\ 0\rangle \quad \text{and} \quad \mathbf{r}_y = \langle 0,\ 1,\ -y/2\rangle$$

Hence,

$$\mathbf{r}_x \times \mathbf{r}_y = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 0 & 0 \\ 0 & 1 & -y/2 \end{vmatrix} = (y/2)\,\mathbf{j} + \mathbf{k} \quad \text{and} \quad \|\mathbf{r}_x \times \mathbf{r}_y\| = \sqrt{1 + y^2/4}.$$

Therefore, letting $D = \{(x, y),\ 0 \le x \le 1,\ -4 \le y \le 4\}$, we have

$$\begin{aligned} \iint_S x\sqrt{y^2+4}\, dS &= \iint_D x\sqrt{y^2+4}\, \|\mathbf{r}_x \times \mathbf{r}_y\|\, dA = \int_D x\sqrt{y^2+4}\,\sqrt{1 + y^2/4}\, dA \\ &= \int_0^1 x\, dx \int_{-4}^4 \frac{y^2+4}{2}\, dy = \frac{1}{2}\left[\frac{y^3}{6} + 2\,y\right]_{-4}^4 \\ &= \frac{56}{3}. \end{aligned}$$

3 Section 16

16, Problem 1

The surface is parametrized by

$$\mathbf{r}(x,y) = \langle x,\, y,\, x^2+y^2\rangle, \quad \text{for } -1 \le x \le 1,\ -1 \le y \le 1.$$

We have

$$\mathbf{r}_x = \langle 1,\, 0,\, 2\,x\rangle \quad \text{and} \quad \mathbf{r}_y = \langle 0,\, 1,\, 2\,y\rangle.$$

Hence,

$$\mathbf{r}_x \times \mathbf{r}_y = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 0 & 2\,x \\ 0 & 1 & 2\,y \end{vmatrix} = (-2\,x)\,\mathbf{i} + (-2\,y)\,\mathbf{j} + \mathbf{k}$$

Note that since the z-coordinate of the normal is positive, the normal points upwards and our parametrization gives the correct orientation. Therefore, letting $D = \{(x,y),\ -1 \le x \le 1,\ -1 \le y \le 1\}$, we have

$$\begin{aligned} \iint_S \left[z\,\mathbf{i} + x^2\,\mathbf{k}\right] \cdot d\mathbf{S} &= \iint_D \left[(x^2+y^2)\,\mathbf{i} + x^2\,\mathbf{k}\right] \cdot \mathbf{r}_x \times \mathbf{r}_y\,dA \\ &= \iint_D \left[(x^2+y^2)\,\mathbf{i} + x^2\,\mathbf{k}\right] \cdot \left[(-2\,x)\,\mathbf{i} + (-2\,y)\,\mathbf{j} + \mathbf{k}\right]\,dA \\ &= \int_{-1}^{1}\int_{-1}^{1} -2\,x^3 - 2\,x\,y^2 + x^2\,dx\,dy = 2\int_{-1}^{1} x^2\,dx = \frac{4}{3}. \end{aligned}$$

16, Problem 4

The surface is the graph of the function $z = x^2 + y^2$ defined for $x^2 + y^2 \le 1$. It is thus parametrized by

$$\mathbf{r}(x,y) = \langle x,\, y,\, x^2+y^2\rangle, \quad \text{for } x^2+y^2 \le 1.$$

As in Problem 1, $\mathbf{r}_x \times \mathbf{r}_y = (-2\,x)\,\mathbf{i} + (-2\,y)\,\mathbf{j} + \mathbf{k}$. Since the normal points outward (i.e. away from the z-axis), this parametrization give sthe correct orientation. Letting $D = \{(x,y),\ x^2+y^2 \le 1\}$, we have

$$\begin{aligned} \iint_S \left[(4\,x)\,\mathbf{i} + (4\,y)\,\mathbf{j} + 2\,\mathbf{k}\right] \cdot d\mathbf{S} &= \iint_D \left[(4\,x)\,\mathbf{i} + (4\,y)\,\mathbf{j} + 2\,\mathbf{k}\right] \cdot \mathbf{r}_x \times \mathbf{r}_y\,dA \\ &= \iint_D \left[(4\,x)\,\mathbf{i} + (4\,y)\,\mathbf{j} + 2\,\mathbf{k}\right] \cdot \left[(-2\,x)\,\mathbf{i} + (-2\,y)\,\mathbf{j} + \mathbf{k}\right]\,dA \\ &= \iint_D 2 - 8\,(x^2+y^2)\,dA. \end{aligned}$$

Passing to polar coordinates, this last integral becomes

$$\int_0^{2\pi}\int_0^1 (2 - 8\,r^2)\,r\,dr\,d\theta = 2\,\pi\int_0^1 2\,r - 8\,r^3\,dr = 2\,\pi\,\left[r^2 - 2\,r^4\right]_0^1 = -2\,\pi.$$

Additional problems

Problem 1

Let S be the surface parametrized by

$$\mathbf{r}(u,v) = (v\,\cos u)\,\mathbf{i} + (v\,\sin u)\,\mathbf{j} + (1+v)\,\mathbf{k}, \quad 0 \le u \le 2\,\pi, \quad 1 \le v \le 2.$$

Find the surface area of S.

Solution. We have

$$\mathbf{r}_u = (-v\,\sin u)\,\mathbf{i} + (v\,\cos u)\,\mathbf{j} \quad \text{and} \quad \mathbf{r}_v = (\cos u)\,\mathbf{i} + (\sin u)\,\mathbf{j} + \mathbf{k}.$$

Thus,

$$\mathbf{r}_u \times \mathbf{r}_v = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -v\,\sin u & v\,\cos u & 0 \\ \cos u & \sin u & 1 \end{vmatrix} = (v\,\cos u)\,\mathbf{i} + (v\,\sin u)\,\mathbf{j} + (-v)\,\mathbf{k}$$

and

$$\|\mathbf{r}_u \times \mathbf{r}_v\| = \sqrt{v^2\,(\cos^2 u + \sin^2 u) + v^2} = \sqrt{2}\,|v|.$$

Therefore, letting $D = \{(u,v),\ 0 \le u \le 2\,\pi,\ 1 \le v \le 2\}$, we obtain

$$A(S) = \iint_S 1\,dS = \iint_D \|\mathbf{r}_u \times \mathbf{r}_v\|\,dA = \int_0^{2\,\pi} \int_1^2 \sqrt{2}\,v\,dv\,du = 2\,\pi\,\sqrt{2} \int_1^2 v\,dv = 3\,\pi\,\sqrt{2}.$$

Problem 2

Let S be the part of the paraboloid $x^2 + y^2 - z^2 = 1$ A parametrization for S is given by

$$\mathbf{r}(u,v) = (\cos u - v\,\sin u)\,\mathbf{i} + (\sin u + v\,\cos u)\,\mathbf{j} + (v)\,\mathbf{k}, \quad 0 \le u \le 2\,\pi, \quad 1 \le v \le 2.$$

Compute the surface integral $\iint_S z\,dS$.

Solution. We compute

$$\mathbf{r}_u = (-\sin u - v\,\cos u)\,\mathbf{i} + (\cos u - v\,\sin u)\,\mathbf{j} \quad \text{and} \quad \mathbf{r}_v = (-\sin u)\,\mathbf{i} + (\cos u)\,\mathbf{j} + \mathbf{k}.$$

and

$$\begin{aligned} \mathbf{r}_u \times \mathbf{r}_v &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ (-\sin u - v\,\cos u) & (\cos u - v\,\sin u) & 0 \\ -\sin u & \cos u & 1 \end{vmatrix} \\ &= (\cos u - v\,\sin u)\,\mathbf{i} + (\sin u + v\,\cos u)\,\mathbf{j} + (-v)\,\mathbf{k}. \end{aligned}$$

Thus,

$$\|\mathbf{r}_u \times \mathbf{r}_v\| = \sqrt{(\cos u - v\,\sin u)^2 + (\sin u + v\,\cos u)^2 + v^2} = \sqrt{1 + 2\,v^2}.$$

Therefore, letting $D = \{(u, v),\ 0 \le u \le 2\,\pi,\ 1 \le v \le 2\}$, we obtain

$$\iint_S z\,dS = \iint_D v\,\|\mathbf{r}_u \times \mathbf{r}_v\|\,dA = \int_0^{2\,\pi} \int_1^2 v\,\sqrt{1 + 2\,v^2}\,dv\,du$$
$$= 2\,\pi \left[\frac{(1 + 2\,v^2)^{3/2}}{6}\right]_1^2 = \frac{\pi}{3}\,\left(9^{3/2} - 3^{3/2}\right)$$
$$= \pi\left(9 - \sqrt{3}\right).$$

Problem 3

Let $\mathbf{F}(x, y, z) = (-z)\,\mathbf{i} + (2\,x)\,\mathbf{j} + (y)\,\mathbf{k}$. Let C be a circle of radius R lying in the plane $2\,x + y + 3\,z = 6$. What are the possible values of $\oint_C \mathbf{F}\,d\mathbf{r}$?

Solution. We apply Stokes' theorem. We first compute

$$\nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ -z & 2\,x & y \end{vmatrix} = \mathbf{i} - \mathbf{j} + 2\,\mathbf{k}.$$

The plane $2\,x + y + 3\,z = 6$ can be parametrized by

$$\mathbf{r}(x, y) = \langle x,\ y,\ 2 - 2\,x/3 - y/3\rangle.$$

We have

$$\mathbf{r}_x = \langle 1,\ 0,\ -2/3\rangle \quad \text{and} \quad \mathbf{r}_y = \langle 0,\ 1,\ -1/3\rangle.$$

Hence,

$$\mathbf{r}_x \times \mathbf{r}_y = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 0 & -2/3 \\ 0 & 1 & -1/3 \end{vmatrix} = (2/3)\,\mathbf{i} + (1/3)\,\mathbf{j} + \mathbf{k}$$

If D is the disk in the plane $2\,x + y + 3\,z = 6$ enclosed by the circle C and R is the projection of D onto the x, y plane (so that D is the image of R under the mapping $\mathbf{r}$), we have, using Stokes' theorem, if C and D have compatible orientations, that

$$\oint_C \mathbf{F}\,d\mathbf{r} = \iint_D \nabla \times \mathbf{F} \cdot d\mathbf{S} = \iint_R \langle 1,\ -1,\ 2\rangle \cdot \langle 2/3,\ 1/3,\ 1\rangle\,dA = \frac{7}{3}\,A(R).$$

Note that

$$A(D) = \iint_D dS = \iint_R \|\mathbf{r}_x \times \mathbf{r}_y\|\,dA = \iint_R \frac{\sqrt{14}}{3}\,dA = \frac{\sqrt{14}}{3}\,A(R).$$

Thus,

$$\oint_C \mathbf{F}\, d\mathbf{r} = \pm \frac{7}{3} \frac{3}{\sqrt{14}}\, A(D) = \pm \frac{7}{\sqrt{14}}\, \pi R^2 = \pm \frac{\sqrt{14}}{2}\, \pi R^2,$$

where the minus sign would appear when the orientation of C and D are not compatible.

Problem 4

Let $\mathbf{F}(x, y, z) = (-y)\, \mathbf{i} + (x)\, \mathbf{j} + (z)\, \mathbf{k}$. Let C be a curve parametrized by

$$\mathbf{r}(t) = (\cos t)\, \mathbf{i} + (\sin t)\, \mathbf{j} + (\sin^2 t)\, \mathbf{k}, \quad 0 \le t \le 2\pi,$$

oriented counterclockwise as viewed from above and let S be the part of the surface $z = y^2$ inside the cylinder $x^2 + y^2 = 1$ oriented such that the normal is pointing upward. Note that C is the boundary of S and that their orientations are compatible. Compute the integral $\oint_C \mathbf{F}\, d\mathbf{r}$ directly and also using Stoke' theorem.

Solution. We first compute the integral directly. We have

$$\mathbf{r}'(t) = (-\sin t)\, \mathbf{i} + (\cos t)\, \mathbf{j} + (2\, \sin t\, \cos t)\, \mathbf{k}$$

Thus,

$$\oint_C \mathbf{F}\, d\mathbf{r} = \int_0^{2\pi} \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t)\, dt = \int_0^{2\pi} \langle -\sin t,\, \cos t,\, \sin^2 t \rangle \cdot \langle -\sin t,\, \cos t,\, 2\, \sin t\, \cos t \rangle\, dt$$

$$= \int_0^{2\pi} 1 + 2\, \sin^3 t\, \cos t\, dt = \left[t + \frac{\sin^4 t}{2} \right]_0^{2\pi} = 2\pi.$$

We then compute the integral using Stoke' theorem. We first compute

$$\nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ -y & x & z \end{vmatrix} = 2\, \mathbf{k}.$$

The surface S can be parametrized by

$$\mathbf{r}(x, y) = \langle x,\, y,\, y^2 \rangle, \quad \text{where} \quad x^2 + y^2 \le 1.$$

We have

$$\mathbf{r}_x = \langle 1,\, 0,\, 0 \rangle \quad \text{and} \quad \mathbf{r}_y = \langle 0,\, 1,\, 2\, y \rangle.$$

Hence,

$$\mathbf{r}_x \times \mathbf{r}_y = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 0 & 0 \\ 0 & 1 & 2\, y \end{vmatrix} = (-2\, y)\, \mathbf{j} + \mathbf{k}$$

Therefore, letting $D = \{(x, y),\ x^2 + y^2 \le 1\}$, we have

$$\oint_C \mathbf{F}\, d\mathbf{r} = \iint_S \nabla \times \mathbf{F} \cdot d\mathbf{S} = \iint_D \langle 0,\, 0,\, 2 \rangle \cdot \langle 0,\, -2\, y,\, 1 \rangle\, dA$$

$$= \iint_D 2\, dA = \int_0^{2\pi} \int_0^1 2\, r\, dr\, d\theta$$

$$= 2\pi.$$