HOMEWORK #3: VECTOR CALCULUS: LINE AND DOUBLE INTEGRALS

Due: one minute after 11:59pm on March 1

Instructions:

- The assignment consists of *four* questions worth, respectively, 2, 3, 2, and 3 points.
- Submit your assignment *electronically* to the Email address specific to your last name as indicated on the course website; the file containing your assignment must be named `Name_0XXXXXX_hwN.m`, where "`Name`" is your last name, "`XXXXXX`" is your student ID number, and "`N`" is the consecutive number of the assignment; hardcopy submissions will not be accepted.
- It is obligatory to use the *current* MATLAB template file available at `http://www.math.mcmaster.ca/bprotas/MATH2ZZ3a/template.m`; submissions non compliant with this template will not be accepted.
- Make sure to enter your name and student I.D. number in the appropriate section of the template.
- Late submissions and submissions which do not comply with these guidelines will not be accepted.
- All graphs should contain suitable titles and legends.
- Reference:
  1. "**Numerical Mathematics**" by M. Grasselli and D. Pelinovsky (Jones and Bartlett, 2008), sections 7.4, 7.6, 7.7.
  2. "**Advanced Engineering Mathematics**" by D.G. Zill and M.R. Cullen (Jones and Bartlett, 3rd edition), sections 9.8-9.10.

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1. Plot the curve

$$\mathbf{s}(t) = \begin{bmatrix} 3\cos(t) \\ -2\sin(t) \\ t \end{bmatrix}$$

for $0 \le t \le 4\pi$ with the step size $\Delta t = 0.1$ and its tangent vectors with the step size $\Delta t = 0.5$ on the same graph using the MATLAB commands `plot3` and `quiver3`. The graph should appear as `Figure 1`.
Calculate the length of the curve $\mathbf{s}(t)$ for $0 \le t \le 2\pi$ using the definition of the *arc length* of a continuously differentiable path $\mathbf{s}$

$$L(\mathbf{s}) = \int_a^b |\dot{\mathbf{s}}(t)|dt$$

and the MATLAB function `quad`. Save the result in the variable `Answer1`. Obtain also another approximation of $L(\mathbf{s})$ by adding the elementary lengths $|\mathbf{s}(t_i) - \mathbf{s}(t_{i-1})|$ with $t_i - t_{i-1} = \frac{2\pi}{100}$. Save this result in the variable `Answer2`.

2. Consider the electric field

$$\mathbf{E}(\mathbf{x}) = \frac{q\mathbf{x}}{4\pi\varepsilon_0\|\mathbf{x}\|^3}, \tag{1}$$

where $\mathbf{x} = [x_1,\ x_2,\ x_3]^T$. Using the MATLAB function `quiver3` plot the vector field $\mathbf{E}(\mathbf{x})$ for $x_1, x_2, x_3 \in [-1, 1]$ with the step sizes $\Delta x_1 = \Delta x_2 = \Delta x_3 = 0.2$. On the same figure plot also the divergence of $\mathbf{E}(\mathbf{x})$ in the form of a slice through the plane $x = 0$ (use the MATLAB function `slice`). For computations use the values $q = 10^{-10}$ and $\varepsilon_0 = 8.85 \cdot 10^{-12}$. The graph should appear as `Figure 2`.
Using the MATLAB function `quad` calculate the line integral of $\mathbf{E}$ along the line segment connecting the points $(1,0,0)$ and $(0,0,1)$. Also, using the same technique, calculate the line integral of $\mathbf{E}$ along the arc of the circle connecting the points $(1,0,0)$ and $(0,0,1)$ over a sphere of radius one centered at the origin. Save the results in the variables `Answer3` and `Answer4`, respectively.
Find a scalar function $f : \mathbb{R}^3 \to \mathbb{R}$ satisfying the relation $\mathbf{E} = \nabla f$ and compare the results of the two previous calculations with the difference $f(0,0,1) - f(1,0,0)$. Save your conclusion in the variable `Answer5` (as text).

3. Use MATLAB to verify *Gauss' law* for the flux of electric field (1) which states that this flux equals q/ε_0, i.e.,

$$\int\!\!\int_S \mathbf{E} \cdot d\mathbf{s} = \frac{q}{\varepsilon_0}. \tag{2}$$

For your computations, use the values $q = 10^{-10}$ and $\varepsilon_0 = 8.85 \cdot 10^{-12}$ and assume that S is a cube centered at the origin with the sides of length $l = 1$. Save the results for the left–hand and right–hand sides of (2) in the variables `Answer6` and `Answer7`, respectively.

4. Calculate the area of the torus given by

$$\mathbf{S}(u,v) = \begin{bmatrix} (5+2\sin(u))\cos(v) \\ (5+2\sin(u))\sin(v) \\ 2\cos(u) \end{bmatrix} \tag{3}$$

for $0 \le u \le 2\pi$ and $0 \le v \le 2\pi$ using the MATLAB command `dblquad` and the formula

$$A(\mathbf{S}) = \int_c^d \int_a^b |\mathbf{t}_u \times \mathbf{t}_v| du\, dv,$$

where

$$\mathbf{t}_u(u_0,v_0) = \left.\frac{\partial \mathbf{S}(u,v_0)}{\partial u}\right|_{u=u_0}, \quad \mathbf{t}_v(u_0,v_0) = \left.\frac{\partial \mathbf{S}(u_0,v)}{\partial v}\right|_{v=v_0}$$

are tangent vectors at the point $\mathbf{S}(u_0,v_0)$. Save the result in the variable `Answer8`. The graph of torus (3) obtained with the step sizes $\Delta u = \Delta v = 0.2$ should appear as `Figure 3`.