

PART III

Review of (Abstract) Approximation Theory

*Although this may seem a paradox,
all exact science is dominated by the idea of approximation.*

— Bertrand Russell (1872–1970)

INNER PRODUCTS, UNITARY SPACES, HILBERT SPACES

- Consider a real or complex linear space \mathbf{V} ; A **SCALAR PRODUCT** is real or complex number (x, y) associated with the elements $x, y \in \mathbf{V}$ with the following properties:
 - (x, x) is real, $(x, x) \geq 0$, $(x, x) = 0$ only if $x = 0$,
 - $(x, y) = \overline{(y, x)}$,
 - $(\alpha_1 x_1 + \alpha_2 x_2, y) = \alpha_1 (x_1, y) + \alpha_2 (x_2, y)$
- A normed space \mathbf{V} is said to be **UNITARY** if its norm and scalar product are connected via the following relation: $\|x\| = (x, x)^{1/2}$
- A complete unitary space \mathbf{H} is called a **HILBERT SPACE**

ORTHOGONALITY

- Two elements x and y of a Hilbert space \mathbf{V} are said to be mutually **ORTHOGONAL** ($x \perp y$) if $(x, y) = 0$. A countable set of elements $x_1, x_2, \dots, x_k, \dots$ is said to be **ORTHONORMAL** (or to form **AN ORTHONORMAL SYSTEMS**) if $(x_i, x_j) = \delta_{ij}$
- The following properties hold:
 - $x \perp 0$ for all $x \in \mathbf{V}$
 - $x \perp x$ only if $x = 0$
 - if $x \perp \mathbf{A}$, i.e., $x \perp y$ for all $y \in \mathbf{A} \subseteq \mathbf{V}$, then x is also orthogonal to the linear hull $\mathcal{L}(\mathbf{A})$
 - if $x \perp y_n$ ($n = 1, 2, \dots$) and $y_n \rightarrow y$, then $x \perp y$
 - if \mathbf{A} is dense in \mathbf{V} and $x \perp \mathbf{A}$, then $x = 0$
- **SCHMIDT ORTHOGONALIZATION** — Let \mathbf{A} be a set of countably many linearly independent elements $x_1, x_2, \dots, x_k, \dots$ of a Hilbert space \mathbf{H} . Then there is an orthonormal system $\mathbf{F} = \{e_i \in \mathbf{V} : (e_i, e_j) = \delta_{ij}\}$, such that the linear hulls of $\mathbf{A}_k = \{x_j : j = 1, \dots, k\}$ and $\mathbf{F}_k = \{e_j : j = 1, \dots, k\}$ are the same for all k .

APPROXIMATION IN HILBERT SPACES (I)

- Let $\{e_1, e_2, \dots\}$ be an orthonormal system in a Hilbert space \mathbf{H} and let \mathbf{H}_k be the linear hull of $\{e_1, \dots, e_k\}$. Then for every $x \in \mathbf{H}$ the element $a = \sum_{j=1}^k (x, e_j) e_j \in \mathbf{H}_k$ has the property that $\|x - a\| \leq \|x - y\|$ for all $y \in \mathbf{H}_k$. The numbers (x, e_j) are called **THE FOURIER COEFFICIENTS** relative to the orthonormal system $\{e_1, e_2, \dots\}$. Furthermore, from $\|x - a\|^2 \geq 0$ follows the **BESSEL INEQUALITY** :

$$\sum_{j=1}^k |(x, e_j)|^2 \leq (x, x)$$

- If \mathbf{A} is a given set in a Hilbert space \mathbf{H} , then

$$\mathbf{A}^\perp = \{x : (x, a) = 0 \text{ for all } a \in \mathbf{A}\}$$

is a closed linear subspace of \mathbf{H} . It is, therefore, itself a Hilbert space and is called **THE ORTHOGONAL COMPLEMENT OF \mathbf{A}**

APPROXIMATION IN HILBERT SPACES (II)

- If \mathbf{H}_1 is a closed linear subspace of a Hilbert space \mathbf{H} and \mathbf{H}_2 is its orthogonal complement, then every $x \in \mathbf{H}$ can be uniquely represented in the form

$$x = x_1 + x_2, \quad (x_1 \in \mathbf{H}_1, x_2 \in \mathbf{H}_2)$$

One writes $\mathbf{H} = \mathbf{H}_1 \oplus \mathbf{H}_2$ and calls \mathbf{H} an orthogonal sum of \mathbf{H}_1 and \mathbf{H}_2 . Since

$$\|x - x_1\| = \rho(x, \mathbf{H}_1) = \inf_{y_1 \in \mathbf{H}_1} \{\|x - y_1\|\},$$

$$\|x - x_2\| = \rho(x, \mathbf{H}_2) = \inf_{y_2 \in \mathbf{H}_2} \{\|x - y_2\|\},$$

one calls x_1 and x_2 the **ORTHOGONAL PROJECTIONS** of x on \mathbf{H}_1 and \mathbf{H}_2 , respectively.

APPROXIMATION IN HILBERT SPACES (III)

- Let $\{e_1, e_2, \dots\}$ be a countable orthonormal system in a Hilbert space \mathbf{H} . By Bessel inequality, the series $\sum_{j=1}^{\infty} (x, e_j) e_j = \lim_{n \rightarrow \infty} \sum_{j=1}^n (x, e_j) e_j$ defines an element of \mathbf{H} for every $x \in \mathbf{H}$. This is called **THE FOURIER SERIES OF x**
- The partial sum $s_n = \sum_{j=1}^n (x, e_j) e_j$ is the orthogonal projection of x on the subspace $\mathbf{H}_n = \mathcal{L}(\{e_1, \dots, e_n\})$. One has $\|s_n\|^2 = \sum_{j=1}^n |(x, e_j)|^2$
- If the system $\{e_1, \dots, e_k, \dots\}$ is complete in \mathbf{H} , i.e., $\overline{\mathcal{L}(\{e_1, \dots, e_k, \dots\})} = \mathbf{H}$, then the Fourier series for any $x \in \mathbf{H}$ converges to x
- An orthonormal system is said to be **CLOSED** if **THE PARCEVAL EQUATION**

$$\sum_{j=1}^{\infty} |(x, e_j)|^2 = \|x\|^2$$

holds for every $x \in \mathbf{H}$. An orthonormal system is closed IFF it is complete.

- An orthonormal system in a separable Hilbert space is at most countable

APPROXIMATION IN HILBERT SPACES (IV)

- Statement of a **GENERAL APPROXIMATION PROBLEM IN A HILBERT SPACE \mathbf{H}** — consider a fixed element $f \in \mathbf{H}$ and $\mathbf{G}_n \subseteq \mathbf{H}$ which is a finite-dimensional subspace of \mathbf{H} (with the same norm). Want to find an element $\hat{g} \in \mathbf{G}_n$ such that

$$D(f, \mathbf{G}_n, \|\cdot\|) \triangleq \inf_{\mathbf{g} \in \mathbf{G}_n} \{\|\mathbf{f} - \mathbf{g}\|\} = \|\mathbf{f} - \hat{\mathbf{g}}\|$$

The element \hat{g} is called **THE BEST APPROXIMATION** and the number $D(f, \mathbf{G}_n, \|\cdot\|)$ is called **THE DEFECT**.

- Issues:
 - Does the best approximation \hat{g} exist?
 - Can \hat{g} be uniquely determined?
 - How can \hat{g} be computed?

APPROXIMATION IN HILBERT SPACES (V)

- The approximation problem in a Hilbert space \mathbf{H} has a unique solution \hat{g} for which $(\hat{g} - f, h) = 0$ holds for all $h \in \mathbf{G}_n$. If $\{e_1, \dots, e_n\}$ is a basis of \mathbf{G}_n , then

$$\hat{g} = \sum_{j=1}^n c_j^{(n)} e_j$$

with

$$\sum_{j=1}^n c_j^{(n)} (e_j, e_k) = (f, e_k), \quad j = 1, \dots, n \quad (\star)$$

and

$$\|f - \hat{g}\|^2 = (f - \hat{g}, f - \hat{g}) = \|f\|^2 - \sum_{j=1}^n c_j^{(n)} (e_j, f)$$

- Thus, the Fourier coefficients $c_j^{(n)}$ $j = 1, \dots, n$ can be calculated by solving an algebraic system (\star) with the Hermitian, positive-definite matrix $A_{jk} = (e_j, e_k)$ (the so called **GRAM MATRIX**).
- If the basis $\{e_1, \dots, e_n\}$ is orthogonal, the system becomes decoupled and the Fourier coefficients can be calculated simply as $c_k^{(n)} = (f, e_k)$

APPROXIMATION IN HILBERT SPACES (VI) RATES OF CONVERGENCE

- Assume that $c_j, j = 1, 2, \dots$ are the Fourier coefficients related to an approximation of some function $f = \sum_{j=1}^n c_j e_j$
- The **RATE OF CONVERGENCE** of this approximation is:
 - **ALGEBRAIC** with order k if for $j \gg 1$

$$\lim_{j \rightarrow \infty} |c_j| j^k < \infty, \quad \text{or, equivalently, } |c_j| \sim O[j^{-k}]$$

- **EXPONENTIAL OR SPECTRAL** with index r if for **ANY** $j > 0$

$$\lim_{j \rightarrow \infty} |c_j| j^k < \infty, \quad \text{or, equivalently, } |c_j| \sim O[\exp(-qj^r)], \quad r, q \in \mathbb{R}^+$$

spectral convergence can be:

- * **SUBGEOMETRIC** when $r < 1$,
- * **GEOMETRIC** when $r = 1$, and
- * **SUPERGEOMETRIC** otherwise