

## PART III

### Review of (Abstract) Approximation Theory

*Although this may seem a paradox,  
all exact science is dominated by the idea of approximation.*

— Bertrand Russell (1872–1970)

## INNER PRODUCTS, UNITARY SPACES, HILBERT SPACES

- Consider a real or complex linear space  $\mathbf{V}$ ; A **SCALAR PRODUCT** is real or complex number  $(x,y)$  associated with the elements  $x,y \in \mathbf{V}$  with the following properties:
  - $(x,x)$  is real,  $(x,x) \geq 0$ ,  $(x,x) = 0$  only if  $x = 0$ ,
  - $(x,y) = \overline{(y,x)}$ ,
  - $(\alpha_1 x_1 + \alpha_2 x_2, y) = \alpha_1 (x_1, y) + \alpha_2 (x_2, y)$
- A normed space  $\mathbf{V}$  is said to be **UNITARY** if its norm and scalar product are connected via the following relation:  $\|x\| = (x,x)^{1/2}$
- A complete unitary space  $\mathbf{H}$  is called a **HILBERT SPACE**

## ORTHOGONALITY

- Two elements  $x$  and  $y$  of a Hilbert space  $\mathbf{V}$  are said to be mutually **ORTHOGONAL** ( $x \perp y$ ) if  $(x,y) = 0$ . A countable set of elements  $x_1, x_2, \dots, x_k, \dots$  is said to be **ORTHONORMAL** (or to form **AN ORTHONORMAL SYSTEMS**) if  $(x_i, x_j) = \delta_{ij}$
- The following properties hold:
  - $x \perp 0$  for all  $x \in \mathbf{V}$
  - $x \perp x$  only if  $x = 0$
  - if  $x \perp \mathbf{A}$ , i.e.,  $x \perp y$  for all  $y \in \mathbf{A} \subseteq \mathbf{V}$ , then  $x$  is also orthogonal to the linear hull  $\mathcal{L}(\mathbf{A})$
  - if  $x \perp y_n$  ( $n = 1, 2, \dots$ ) and  $y_n \rightarrow y$ , then  $x \perp y$
  - if  $\mathbf{A}$  is dense in  $\mathbf{V}$  and  $x \perp \mathbf{A}$ , then  $x = 0$
- SCHMIDT ORTHOGONALIZATION** — Let  $\mathbf{A}$  be a set of countably many linearly independent elements  $x_1, x_2, \dots, x_k, \dots$  of a Hilbert space  $\mathbf{H}$ . Then there is an orthonormal system  $\mathbf{F} = \{e_i \in \mathbf{V} : (e_i, e_j) = \delta_{ij}\}$ , such that the linear hulls of  $\mathbf{A}_k = \{x_j : j = 1, \dots, k\}$  and  $\mathbf{F}_k = \{e_j : j = 1, \dots, k\}$  are the same for all  $k$ .

## APPROXIMATION IN HILBERT SPACES (I)

- Let  $\{e_1, e_2, \dots\}$  be an orthonormal system in a Hilbert space  $\mathbf{H}$  and let  $\mathbf{H}_k$  be the linear hull of  $\{e_1, \dots, e_k\}$ . Then for every  $x \in \mathbf{H}$  the element  $a = \sum_{j=1}^k (x, e_j) e_j \in \mathbf{H}_k$  has the property that  $\|x - a\| \leq \|x - y\|$  for all  $y \in \mathbf{H}_k$ . The numbers  $(x, e_j)$  are called **THE FOURIER COEFFICIENTS** relative to the orthonormal system  $\{e_1, e_2, \dots\}$ . Furthermore, from  $\|x - a\|^2 \geq 0$  follows the **BESSEL INEQUALITY** :

$$\sum_{j=1}^k |(x, e_j)|^2 \leq (x, x)$$

- If  $\mathbf{A}$  is a given set in a Hilbert space  $\mathbf{H}$ , then

$$\mathbf{A}^\perp = \{x : (x, a) = 0 \text{ for all } a \in \mathbf{A}\}$$

is a closed linear subspace of  $\mathbf{H}$ . It is, therefore, itself a Hilbert space and is called **THE ORTHOGONAL COMPLEMENT OF  $\mathbf{A}$**

## APPROXIMATION IN HILBERT SPACES (II)

- If  $\mathbf{H}_1$  is a closed linear subspace of a Hilbert space  $\mathbf{H}$  and  $\mathbf{H}_2$  is its orthogonal complement, then every  $x \in \mathbf{H}$  can be uniquely represented in the form

$$x = x_1 + x_2, \quad (x_1 \in \mathbf{H}_1, x_2 \in \mathbf{H}_2)$$

One writes  $\mathbf{H} = \mathbf{H}_1 \oplus \mathbf{H}_2$  and calls  $\mathbf{H}$  an orthogonal sum of  $\mathbf{H}_1$  and  $\mathbf{H}_2$ . Since

$$\|x - x_1\| = \rho(x, \mathbf{H}_1) = \inf_{y_1 \in \mathbf{H}_1} \{\|x - y_1\|\},$$

$$\|x - x_2\| = \rho(x, \mathbf{H}_2) = \inf_{y_2 \in \mathbf{H}_2} \{\|x - y_2\|\},$$

one calls  $x_1$  and  $x_2$  the **ORTHOGONAL PROJECTIONS** of  $x$  on  $\mathbf{H}_1$  and  $\mathbf{H}_2$ , respectively.

## APPROXIMATION IN HILBERT SPACES (III)

- Let  $\{e_1, e_2, \dots\}$  be a countable orthonormal system in a Hilbert space  $\mathbf{H}$ . By Bessel inequality, the series  $\sum_{j=1}^{\infty} (x, e_j) e_j = \lim_{n \rightarrow \infty} \sum_{j=1}^n (x, e_j) e_j$  defines an element of  $\mathbf{H}$  for every  $x \in \mathbf{H}$ . This is called **THE FOURIER SERIES OF  $x$**
- The partial sum  $s_n = \sum_{j=1}^n (x, e_j) e_j$  is the orthogonal projection of  $x$  on the subspace  $\mathbf{H}_n = \mathcal{L}(\{e_1, \dots, e_n\})$ . One has  $\|s_n\|^2 = \sum_{j=1}^n |(x, e_j)|^2$
- If the system  $\{e_1, \dots, e_k, \dots\}$  is complete in  $\mathbf{H}$ , i.e.,  $\overline{\mathcal{L}(\{e_1, \dots, e_k, \dots\})} = \mathbf{H}$ , then the Fourier series for any  $x \in \mathbf{H}$  converges to  $x$
- An orthonormal system is said to be **CLOSED** if **THE PARCEVAL EQUATION**

$$\sum_{j=1}^{\infty} |(x, e_j)|^2 = \|x\|^2$$

holds for every  $x \in \mathbf{H}$ . An orthonormal system is closed IFF it is complete.

- An orthonormal system in a separable Hilbert space is at most countable

## APPROXIMATION IN HILBERT SPACES (IV)

- Statement of a **GENERAL APPROXIMATION PROBLEM IN A HILBERT SPACE  $\mathbf{H}$**  — consider a fixed element  $f \in \mathbf{H}$  and  $\mathbf{G}_n \subseteq \mathbf{H}$  which is a finite-dimensional subspace of  $\mathbf{H}$  (with the same norm). Want to find an element  $\hat{g} \in \mathbf{G}_n$  such that

$$D(f, \mathbf{G}_n, \|\cdot\|) \triangleq \inf_{g \in \mathbf{G}_n} \{\|f - g\|\} = \|f - \hat{g}\|$$

The element  $\hat{g}$  is called **THE BEST APPROXIMATION** and the number  $D(f, \mathbf{G}_n, \|\cdot\|)$  is called **THE DEFECT**.

- Issues:
  - Does the best approximation  $\hat{g}$  exist?
  - Can  $\hat{g}$  be uniquely determined?
  - How can  $\hat{g}$  be computed?

## APPROXIMATION IN HILBERT SPACES (V)

- The approximation problem in a Hilbert space  $\mathbf{H}$  has a unique solution  $\hat{g}$  for which  $(\hat{g} - f, h) = 0$  holds for all  $h \in \mathbf{G}_n$ . If  $\{e_1, \dots, e_n\}$  is a basis of  $\mathbf{G}_n$ , then

$$\hat{g} = \sum_{j=1}^n c_j^{(n)} e_j$$

with

$$\sum_{j=1}^n c_j^{(n)} (e_j, e_k) = (f, e_k), \quad j = 1, \dots, n \quad (\star)$$

and

$$\|f - \hat{g}\|^2 = (f - \hat{g}, f - \hat{g}) = \|f\|^2 - \sum_{j=1}^n c_j^{(n)} (e_j, f)$$

- Thus, the Fourier coefficients  $c_j^{(n)}$   $j = 1, \dots, n$  can be calculated by solving an algebraic system  $(\star)$  with the Hermitian, positive-definite matrix  $A_{jk} = (e_j, e_k)$  (the so called **GRAM MATRIX**).
- If the basis  $\{e_1, \dots, e_n\}$  is orthogonal, the system becomes decoupled and the Fourier coefficients can be calculated simply as  $c_k^{(n)} = (f, e_k)$

## APPROXIMATION IN HILBERT SPACES (VI) RATES OF CONVERGENCE

- Assume that  $c_j$ ,  $j = 1, 2, \dots$  are the Fourier coefficients related to an approximation of some function  $f = \sum_{j=1}^n c_j e_j$
- The **RATE OF CONVERGENCE** of this approximation is:

- **ALGEBRAIC** with order  $k$  if for  $j \gg 1$

$$\lim_{j \rightarrow \infty} |c_j| j^k < \infty, \text{ or, equivalently, } |c_j| \sim O[j^{-k}]$$

- **EXPONENTIAL OR SPECTRAL** with index  $r$  if for **ANY**  $j > 0$

$$\lim_{j \rightarrow \infty} |c_j| j^k < \infty, \text{ or, equivalently, } |c_j| \sim O[\exp(-qj^r)], \ r, q \in \mathbb{R}^+$$

spectral convergence can be:

- \* **SUBGEOMETRIC** when  $r < 1$ ,
- \* **GEOMETRIC** when  $r = 1$ , and
- \* **SUPERGEOMETRIC** otherwise