HOMEWORK #5

Due: March 28 (Wednesday) by midnight

Instructions:

- The assignment consists of *two* questions worth 5 and 3 points.
- Submit your assignment *electronically* (via Email) to the instructor; hardcopy submissions will not be accepted.
- It is obligatory to use the MATLAB template file available at `http://www.math.mcmaster.ca/~bprotas/MATH3Q03/template.m` (see also the link in the "Computer Programs" section of the course website on the left); submissions non compliant with this template will not be accepted.
- Make sure to enter your name and student I.D. number in the appropriate section of the template.
- Late submissions and submissions which do not comply with these guidelines will not be accepted.

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1. Consider a generalized form of Runge's function

$$f_\alpha(x) = \frac{1}{1+(\alpha x)^2}, \tag{1}$$

where $x \in [-5,5]$ and $\alpha \in (0,1]$ is a parameter. Set $\alpha_k = \left(\frac{4}{5}\right)^k$, where $k = 0,\dots,10$.

   (a) For every value of α_k solve the polynomial interpolation problem for the function $f_{\alpha_k}(x)$ using a *uniform* grid and interpolating polynomials $P_N(x)$ of degrees $N = \{4,8,12,16\}$; for every value of α_k plot the interpolation errors defined as $E(N,\alpha_k) = max_j|f_{\alpha_k}(\xi_j) - P_N(\xi_j)|$, where $\xi_j = -5 + j\cdot 0.1$, $j = 0,\dots,100$ (use a single figure to plot all the errors); what is the value of α_j for which the Runge phenomenon disappears?

   (b) solve the same set of interpolation problems using the *Chebyshev* grid instead of the uniform grid; plot the interpolation errors on a separate figure;

   HINT — use the functions `polyfit` and `polyval` to solve the interpolation problems in both cases.
   (5 points)

2. You are given the *cardinal Whittaker* function $f(x) = \frac{\sin(x)}{x}$ in the interval $\Omega = [-\pi,\pi]$.

   (a) Using the step size $h = 0.2$ calculate approximations to $f'(x)$ in Ω using the following methods:

      i. second–order *one–sided* (forward) differences,
      ii. second–order central differences,
      iii. complex step derivative, i.e., $f'(x) = \frac{\Im(F(x+ih))}{h} + O(h^3)$ where $F(z)$ is a complex analytic extension of $f(x)$, i.e., $F : \mathbb{C} \to \mathbb{C}$ and $F(x) = f(x)$ for $x \in \mathbb{R}$,
      iv. fourth–order central differences.

   Plot the resulting curves using solid lines in different colors and the derivative computed analytically using a dashed line.

(b) Using the four approximate methods mentioned above, calculate for $x_0 = \frac{\pi}{12}$ the relative errors of the derivatives as $\left|\frac{f'_{approx}(x_0) - f'_{exact}(x_0)}{f'_{exact}(x_0)}\right|$ and plot them on a log–log graph as a function of the step size h (for h use the values obtained with `logspace(-10,0,10)`).

(3 points)