HOMEWORK #3

Due: November 12 (Thursday) by midnight

Instructions:

- The assignment consists of *three* questions, worth 3, 3 and 4 points.
- Submit your assignment *electronically* to the Email address `math3q03@math.mcmaster.ca`; hardcopy submissions will not be accepted.
- It is obligatory to use the MATLAB template file available at `http://www.math.mcmaster.ca/bprotas/MATH3Q03/template.m` (see also the link in the "Computer Programs" section of the course website); submissions non compliant with this template will not be accepted.
- All plots should have suitable axis labels and legends.
- Make sure to enter your name and student I.D. number in the appropriate section of the template.
- Late submissions and submissions which do not comply with these guidelines will not be accepted.

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1. You are given the following function

$$F(x,y) = \cos(y)^2(x-1)^4 + 100(1-\cos(y))^2. \tag{1}$$

   (a) Using the command `contourf` plot this function in the domain $D = [-5,5] \times [-5,5]$ using the step size $h = 0.1$; this plot should appear as `figure(1)` and should have a colorbar,

   (b) use Newton's method to find the critical points (x_1^*, y_1^*) and (x_2^*, y_2^*) of the function $F(x,y)$ starting from the following two initial guesses:

$$(x_1^0, y_1^0) = (2,1),$$
$$(x_2^0, y_2^0) = (5,3);$$

   mark the approximations of the two critical points obtained during iterations on `figure(1)` using symbols in different colors for the two cases; determine the type of each of these critical points (i.e., whether it is a local minimum, maximum, or a saddle) and print this information on the screen.

   HINT — critical points of function (1) can be found by solving the vector equation $\boldsymbol{\nabla} F(x,y) = \mathbf{0}$, whereas information about the type of the critical point is contained in the Hessian of the function.
   (3 points)

2. Construct `chebfun` objects representing the function $f(x) = \sin(x)$ for $x \in [0, 2\pi]$ and its polynomial approximations $f_n(x)$ obtained using Taylor series expansions centered at the origin in which $n = 1, \dots, 10$ denotes the number of terms. Plot these objects in `figure(2)` restricting the vertical axis to $[-2, 2]$ and using thick line to represent the function $f(x)$. Compute the approximation error $\max_{0 \le x \le 2\pi} |f(x) - f_n(x)|$ and plot it as a function of the number of terms $n = 1, \dots, 10$ in `figure(3)`. This plot should use the logarithmic scaling for the vertical axis, whereas the expression $\max_{0 \le x \le 2\pi} |\cdot|$ can be evaluated using `chebfun` function `norm(., inf)`.
   (3 points)

3. Your are given the function $f(x) = x\cos(x^2)$ in the interval $\Omega = [-\pi, \pi]$ which is discretized using a uniform mesh with $N + 1$ grid points (take $x_0 = -\pi$ and $x_N = \pi$).

   (a) consider the Vandermonde approach to interpolation and determine the value of N for which the condition number κ (obtained using MATLAB function `cond`) of the interpolation matrix exceeds the threshold $E = 10^6$; write out this value of N,

   (b) calculate the Lagrange interpolating polynomials in the case when $N = 4$ and plot them using different colors and the step size $h = x_{i+1} - x_i = \frac{\pi}{50}$ in `figure(4)`,

   (c) plot the function $f(x)$ together with its interpolants constructed using the Lagrange polynomials with $N = 4$ (in `figure(5)`) and $N = 16$ (in `figure(6)`); all functions should be plotted for $x \in \Omega$ using the same step size h as above; mark the interpolation points $\{x_i, f(x_i)\}_{i=0}^{N}$ with symbols,

   (d) using the Lagrange interpolating polynomials constructed above determine and print out the interpolation error at $x = 0.01$ for $N = 2, 4, \ldots, 10$; repeat this calculation using the function $g(x) = |f(x)|$ and explain the different behavior of the error in the two cases; print your answer on the screen.

   (4 points)