HOMEWORK #5

Due: December 10 (Thursday) by midnight

Instructions:

- The assignment consists of *three* questions, worth 2, 4 and 4 points.
- Submit your assignment *electronically* to the Email address `math3q03@math.mcmaster.ca`; hardcopy submissions will not be accepted.
- It is obligatory to use the MATLAB template file available at `http://www.math.mcmaster.ca/bprotas/MATH3Q03/template.m` (see also the link in the "Computer Programs" section of the course website); submissions non compliant with this template will not be accepted.
- All plots should have suitable axis labels and legends.
- Make sure to enter your name and student I.D. number in the appropriate section of the template.
- Late submissions and submissions which do not comply with these guidelines will not be accepted.

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1. Suppose $n = 10$ and construct Chebyshev grid with $(n+1)$ points. Compute the number

$$S = (\ [\text{sum of the last 5 digits in your student number}] \mod n\ ) \tag{1}$$

and plot the Chebyshev polynomial $T_S(x)$ defined on $[-1, 1]$ together with 6 lowest-degree polynomials $T_m(x)$ which are aliased to $T_S(x)$. This plot should appear as `figure(1)` with the polynomial $T_S(x)$ marked with a thick solid line and the aliases $T_m(x)$ marked with thin solid lines. Mark the intersection points of the graphs of $T_S(x)$ and $T_m(x)$ with small circles. To reveal small-scale features of the aliases, in `figure(2)` show the same data in a region of width $\Delta x = 0.05$ centered around the leftmost Chebyshev point x_i located in the *interior* of the domain $[-1, 1]$. Perform these computations using `chebfun`.
(2 points)

2. You are given the following three functions, all defined for $x \in [-1, 1]$,

$$f(x) = \exp(\cos^2(\pi x)),$$
$$g(x) = \exp\left(\frac{-1}{1-x^4}\right),$$
$$h(x) = \begin{cases} 0, & x \le 0, \\ x^5, & x > 0 \end{cases}.$$

For each of these functions consider polynomial interpolation on the Chebyshev grid with points $x_i = \cos(i\pi/n)$, $0 \le i \le n$, i.e.,

$$\{f, g, h\}(x_i) = p_n(x_i), \qquad 0 \le i \le n$$

and find the smallest number n such that the interpolating polynomial $p_n(x)$ satisfies the condition

$$\frac{\|\{f,g,h\} - p_n\|_\infty}{\|\{f,g,h\}\|_\infty} \le \epsilon, \tag{2}$$

where $\|f\|_\infty = \sup_{-1\le x\le 1} |f(x)|$, $\epsilon = 10^{-1}$ and the notation "$\{f,g,h\}$" means "one of the functions f, g and h". Print the values of n obtained in the three cases on the screen and plot the functions f, g and h together with their lower-degree polynomial interpolants satisfying condition (2) (these plots should appear as `figure(3)`, `figure(4)` and `figure(5)`). Perform these computations using `chebfun`.
(4 points)

3. You are given the *cardinal Whittaker* function $f(x) = \frac{\sin(x)}{x}$ on the interval $\Omega = [-\pi, \pi]$.

   (a) Using the step size $h = 0.4$ calculate approximations to the derivative $f'(x)$ in Ω with the following finite-difference methods:

      i. first-order forward differences,
      ii. second-order central differences,
      iii. complex step derivative, i.e., $f'(x) = \frac{\Im(F(x+ih))}{h} + \mathcal{O}(h^3)$ where $F(z)$ is a complex analytic extension of $f(x)$, i.e., $F : \mathbb{C} \to \mathbb{C}$ and $F(x) = f(x)$ for $x \in \mathbb{R}$,
      iv. fourth-order central differences.

   Plot the resulting curves $f'(x)$ using solid lines in different colours and the derivative computed analytically using a dashed line. This plot should appear as `figure(6)`.

   (b) Using the four methods mentioned above, calculate the relative approximation errors of the derivatives at $x_0 = \frac{\pi}{12}$ as $\left|\frac{f'_{approx}(x_0) - f'_{exact}(x_0)}{f'_{exact}(x_0)}\right|$ and plot them on a log-log graph as a function of the step size h (for h use the values obtained with $\texttt{logspace}(-10, 0, 10)$). This plot should appear as `figure(7)`.

You *need not* use `chebfun` in this problem.
(4 points)