CONVERGENCE THEORY

* FIXED-POINT ITERATIONS $x_{n+1} = g(x_n)$

Using the true solution R, we write an expression for the error in the n-th iteration

$$R - x_{n+1} = R - g(x_n) = g(R) - g(x_n)$$

Multiplying and dividing by $(R - x_n)$

$$R - x_{n+1} = \frac{g(R) - g(x_n)}{R - x_n} (R - x_n)$$

Using the mean-value theorem

$$R - x_{n+1} = g'(\xi_n) \cdot (R - x_n), \text{ where } \xi_n \in [x_n, R]$$

Thus, denoting $e_n = R - x_n$, we have

$$|e_{n+1}| = |g'(\xi_n)| \cdot |e_n|$$

Remarks

* if $|g'(\xi_n)| < 1$ on some interval, the fixed-point iterations will converge for initial values in that interval (this explains why certain rearrangements as $f(x) = 0$ may result in divergent iterations)

* the error at a given iteration is a fraction of the error at the previous iteration $\Rightarrow$ linear convergence

* to ensure convergence, the function $g(x)$ must be contractive, i.e., Lipschitz-continuous with the constant $L < 1$

* Newton's METHOD

Newton's method uses iterations similar to the fixed-point approach

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} = g(x_n)$$

Iterates converge if $|g'(x)| < 1$

Thus

$$|g'(x)| = \left| \frac{f(x) \cdot f''(x)}{[f'(x)]^2} \right| < 1$$ // condition for convergence

Error relation: $R - x_{n+1} = g(R) - g(x_n)$

Taylor-expand $g(x_n)$ about $x = R$ up to the second order

$$\begin{cases} g(x_n) = g(R) + g'(R)(x_n - R) + \frac{g''(\xi)}{2}(x_n - R)^2, & \xi \in [x_n, R] \\ f(R) = 0 \Rightarrow g'(R) = \frac{f(R) \cdot f''(R)}{[f'(R)]^2} = 0 \end{cases}$$

$$\Rightarrow g(x_n) = g(R) + \frac{g''(\xi)}{2}(x_n - R)^2$$

Putting this into the error relation

$$|e_{n+1}| = |g(R) - g(x_n)| = \left| \frac{g''(\xi)}{2} \right| |e_n|^2$$

Remarks

* <u>smallness of the second derivative</u> required for convergence
* the error at a given iteration is the square of the error at the previous iteration — <u>quadratic convergence</u>

SECANT AND FALSE POSITION METHODS

In both cases the consecutive iterates can be written as

$$x_{n+1} = x_n - \frac{f(x_n)}{f(x_n) - f(x_{n-1})}(x_n - x_{n-1}) = g(x_n, x_{n-1})$$

Following a similar approach as above, but with more tedious details, we obtain

$$e_{n+1} = \frac{g(\xi_1, \xi_2)}{2} e_n e_{n-1}, \quad \xi_1, \xi_2 \in [x_n, R]$$

Error is proportional to the two previous errors.

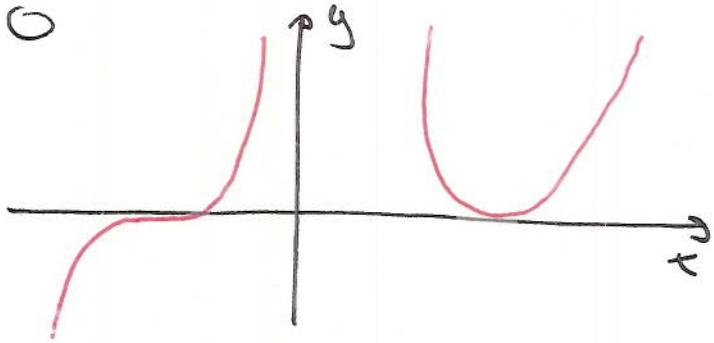
Remark

* It can be shown that the rate of convergence is faster than linear, but slower than quadratic
  $\Rightarrow$ Superlinear convergence

General Remarks

* Higher-order methods (Newton's method) are faster, but come with more stringent requirements on the function $f(x)$ (boundedness of higher derivatives)
* One can combine different methods:
  - start with a slower, but more robust method
  - switch to a faster method close to the root.

* problems occur when multiple roots are present, i.e., when $g(R) = g'(R) = \ldots = 0$



In Newton's method

$$g'(x) = \frac{g(x) \cdot g''(x)}{[g'(x)]^2} \sim \frac{0}{0}$$

doesn't vanish and second-order convergence is lost.

1.3 Numerical Linear Algebra – A Short Review

(MATH 2T03, Grasselli & Pelinovsky Sections 2.3-2.6)

We are interested in the solution of problems of the type

$$Ax = b, \quad x, b \in \mathbb{R}^N, \quad A \in \mathbb{R}^{N \times N}$$

N - large

General remarks about existence of solutions:

* if $\det(A) \neq 0$ a unique solution x exists
  if $b = 0$, $x = 0$

* if $\det(A) = 0$, let $x = x_0 + x_1$
  - $Ax_0 = 0$ has m solutions defined up to multiplicative constants, m – the number of distinct eigenvalues
  - $Ax_1 = b$ – additional solutions exist if $b \perp N(A^T)$ – Fredholm alternative

Cramer's Rule — solution involving determinants (elegant, but useless for large N)

* Elimination Methods

Start with the system below; the goal is to convert it to an upper-triangular form

$$\begin{aligned} 4x_1 - 2x_2 + x_3 &= 15 \quad /\cdot 3 \to \quad /\cdot(-1) \\ -3x_1 - x_2 + 4x_3 &= 8 \quad /\cdot 4 \to (+) \\ x_1 - x_2 + 3x_3 &= 13 \quad /\cdot 4 \to (+) \end{aligned} \Rightarrow$$

$$\Rightarrow \begin{aligned} 4x_1 - 2x_2 - x_3 &= 15 \\ -10x_2 + 19x_3 &= 77 \quad /\cdot 2 \to \\ -2x_2 + 11x_3 &= 37 \quad /\cdot(-10) \to (+) \end{aligned} \Rightarrow \begin{cases} 4x_1 - 2x_2 + x_3 = 15 \\ -10x_2 + 19x_3 = 77 \\ -72x_3 = -216 \end{cases}$$

Given the triangular form, the solution can be found trivially via back-substitution: $x_3 = 3, x_2 = -2, x_1 = 2$

The same goal can be achieved by performing elementary row operations on the corresponding system matrix:

- multiplication of a row by a constant
- adding a multiple of a row to another row
- interchanging the orders of two rows

Problem

- the magnitude of the coefficients grows; this may become an issue for large systems (round-off errors)