Gaussian Elimination — prevent the growth of the coefficients by subtracting $a_{i1}/a_{11} \times R.1$ from the i-th equation when eliminating zeros from the first column

$$\left[\begin{array}{ccc|c} 4 & -2 & 1 & 15 \\ -3 & -1 & 4 & 8 \\ 1 & -1 & 3 & 13 \end{array}\right] \Rightarrow \begin{array}{r} \\ R_2 - (-3/4)R_1 \rightarrow \\ R_3 - (1/4)R_1 \rightarrow \end{array} \left[\begin{array}{ccc|c} 4 & -2 & 1 & 15 \\ 0 & -2.5 & 4.75 & 19.25 \\ 0 & -0.5 & 2.75 & 9.25 \end{array}\right]$$

When performing elimination we must guard against division by zero, or something close to zero.

Pivoting — rearrangement of the equations so as to get the element with the largest magnitude on the diagonal in each step (the pivot element)

Apart from the solution of the algebraic system $Ax = b$, Gaussian elimination also provides the LU-decomposition

$$A = \begin{bmatrix} 1 & 0 & 0 \\ * & 1 & 0 \\ * & * & 1 \end{bmatrix} \begin{bmatrix} & \diagdown & \\ 0 & & \end{bmatrix} = LU$$

It is useful for solving system with one matrix A and multiple RHS vectors b.

Operation count

* Gaussian elimination / LU decomposition — $O(N^3/3)$
  - special case: when the matrix is tri-diagonal, the cost is $O(N)$
* back substitution — $O(N^2)$

$\Rightarrow$ globally $O(N^3/3)$

$$Ax = L\underbrace{Ux}_{y} = Ly = b$$

1) $Ly = b$
2) $Ux = y$

the cost is $O(N^2) \ll O(\frac{N^3}{3})$ for large N

A very efficient solution of an algebraic problem $Ax = b$ is obtained in MATLAB using the "backslash" operator "A\b".
MATLAB function "lu" performs the LU decomposition.

* Matrix Inverse (and associated pathologies)

The matrix inverse A^{-1} is defined as $A \cdot A^{-1} = A^{-1}A = I$.
given a square matrix A the inverse exists iff $\det(A) \neq 0$ (or $\text{rank}(A) = \dim(A) = N$)

If the inverse exists, the solution of system $Ax = b$ can be expressed as $x = A^{-1}b$
The matrix inverse A^{-1} ~~is~~ is useful ~~[illegible]~~ for theoretical analysis, but in practice Gaussian elimination or LU decomposition are much more useful

* Ill-Conditioning

When ~~[illegible]~~ $0 < |\det(A)| \ll 1$, the system is ill-conditioned (almost singular). It does have unique solutions, but they exhibit some pathological properties

Ex

$$\begin{bmatrix} 1.01 & 0.99 \\ 0.99 & 1.01 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2.0 \\ 2.0 \end{bmatrix} \Rightarrow \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1.0 \\ 1.0 \end{bmatrix}$$

$$b = \begin{bmatrix} 2.02 \\ 1.98 \end{bmatrix} \Rightarrow \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2.0 \\ 0 \end{bmatrix}, \quad b = \begin{bmatrix} 1.98 \\ 2.02 \end{bmatrix} \Rightarrow \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 2 \end{bmatrix}$$

Thus, small perturbations of the input b result in very large changes of the output (problem is ill-posed)

Conditioning is characterized by the condition number.
To define it we need the notion of a matrix (operator) norm $\|A\|$ ST

* $\|A\| > 0$, $\|A\| = 0$ iff $A = 0$
* $\|kA\| = |k|\,\|A\|$, $\forall_{k \in \mathbb{C}}$
* $\|A+B\| \leq \|A\| + \|B\|$

} definition of a norm $\|\cdot\|$

Additional property: $\|AB\| \leq \|A\|\,\|B\|$

Vector Norms: $\|x\|_p = \left(\sum_{i=1}^{n} |x_i|^p\right)^{1/p}$, $x \in \mathbb{R}^n$

$$\|x\|_1 = \sum_{i=1}^{n} |x_i|$$

$$\|x\|_2 = \sqrt{\sum_{i=1}^{n} x_i^2} \quad \text{(Euclidean norm)}$$

$$\|x\|_\infty = \max_{1 \leq i \leq n} |x_i|$$