The Bisection Method does not admit a generalization to the multidimensional case ...

Generalization of the Fixed-Point method is straightforward:

$$\bar{x}_{n+1} = \bar{G}(\bar{x}_n), \text{ where } G: \mathbb{R}^n \to \mathbb{R}^n \text{ is}$$

chosen so that $\bar{F}(\bar{x}) = 0 \iff \bar{x} = \bar{G}(\bar{x})$

(this choice of $\bar{G}(\bar{x})$ is still nonunique)

Convergence analysis based on the multidimensional fixed-point theorem:

the contractivity of the mapping $\bar{G}$:

$$\forall_{\bar{x},\bar{y} \in S} \quad \|\bar{G}(\bar{x}) - \bar{G}(\bar{y})\| \leq L \|\bar{x} - \bar{y}\|$$

If $\bar{G}$ is differentiable, it is equivalent to the more easily verified condition

$$\rho(DG(\bar{x})) \leq C < 1 \text{ for } \forall \bar{x} \in S \subset \mathbb{R}^n, \text{ where}$$

$DG(\bar{x})$ – the Jacobian (matrix of partial derivatives) of G at $\bar{x}$

$\rho(\cdot)$ – the spectral radius, i.e., the absolute value of the largest eigenvalue

$$\rho(A) = \max_{1 \leq i \leq m} (|\lambda_i|), \quad m \text{ – number of distinct eigenvalues}$$

Note that when $n=1$ this implies Lipschitz continuity with $L<1$.

* <u>Newton-Raphson method</u> - generalization of Newton's method to a higher dimension.

Taylor expansion of $\bar{F}(\bar{x})=0$ around $\bar{x}=\bar{x}_k$

$$\bar{F}(\bar{x}) = \bar{F}(\bar{x}_k) + DF(\bar{x}_k)(\bar{x}-\bar{x}_k) + O(\|\bar{x}-\bar{x}_k\|^2)$$

$DF(\bar{x}_k)$ - Jacobian of $\bar{F}(\bar{x})$ at $\bar{x}_k$

Setting $\bar{F}(\bar{x}_{k+1})=0 \Rightarrow DF(\bar{x}_k)(\bar{x}_{k+1}-\bar{x}_k) = -\bar{F}(\bar{x}_k)$

Thus

$$\bar{x}_{k+1} = \underbrace{\bar{x}_k - \overbrace{\left[DF(\bar{x}_k)\right]^{-1}\bar{F}(\bar{x}_k)}^{\text{"Newton step"}}}_{\bar{G}(\bar{x}_k)\ \text{(fixed-point iteration)}}$$

<u>Remarks</u>

* Solution of a linear algebraic problem with Jacobian matrix required at every iteration; $DF(\bar{x})$ must be nonsingular

* for convergence, $\rho(DF(\bar{x}))<1$ in the neighbourhood of the root

* quadratic convergence

* Secant method — when the Jacobian DF is not available (or difficult to evaluate), it can be approximated with a matrix D_K satisfying the secant condition:

$$D_k(\bar{x}_k - \bar{x}_{k-1}) = \bar{F}(x_k) - \bar{F}(\bar{x}_{k-1})$$

The choice of D_k is nonunique (unlike in 1D...)

Broyden's method

Specify an "initial guess" D_0 (e.g. $D_0 = I$)

Then iterate

$$D_K(\bar{x}_{k+1} - \bar{x}_k) = -\bar{F}(\bar{x}_k) \qquad \text{(Newton step)}$$

$$D_{k+1} = D_K + \frac{\bar{F}(x_k)\,(\bar{x}_{k+1} - \bar{x}_k)}{\|\bar{x}_{k+1} - \bar{x}_k\|^2} \qquad \text{(update)}$$

for $k = 0, 1, \ldots$

Remarks

* superlinear convergence
* implemented in MATLAB function "fsolve"