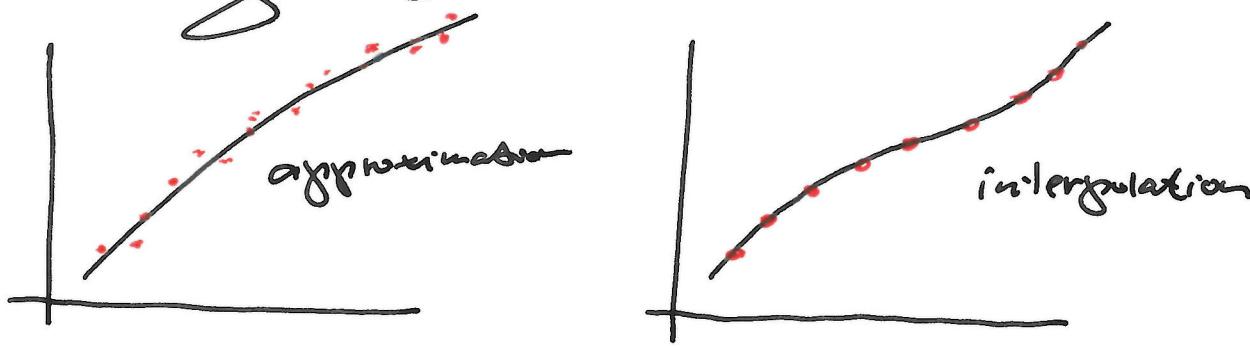


2) INTERPOLATION & APPROXIMATION

(Grasselli & Selenowsky - chapter 5, Trefethen)

Basic definitions



Given a set of data points $\{x_i, y_i\}_{i=1}^N$ and a function $g(x) = \sum_{i=1}^M a_i \phi_i(x)$, where $\{\phi_i\}_{i=1}^M$ are suitably chosen basis functions.

* the interpolation problem is:

Find $\{a_i\}_{i=1}^M$ ST $g(x_i) = y_i, i=1, \dots, M$

* the approximation problem is

Find $\{a_i\}_{i=1}^M$ ST $\sum_{k=1}^N [y_k - g(x_k)]^2 = \min$

Note that usually $N \gg M$.

Questions:

* What are good choices of the basis functions ϕ_i ?

- easy to compute

- fast decrease of error with $M \rightarrow \infty$

* How to determine the expansion coefficients a_1, \dots, a_M ?

- * Estimates of interpolation/approximation errors
- * computational cost

POLYNOMIALS - a first natural choice of basis functions

- * MATLAB convention for indexing coefficients

$$P_n(x) = c_1 x^n + c_2 x^{n-1} + \dots + c_n x + c_{n+1}$$

- * P_n is determined by a coefficient vector $c \in \mathbb{R}^{n+1}$

Hence, polynomials $\approx P_n$ can be identified with a finite-dimensional vector space

$$P_n \in V = \mathbb{R}^{n+1}$$

- * trigonometric functions can also be interpreted as ~~polynomial~~ (complex) polynomials

$$\text{Set } z = e^{ie} = \cos\varphi + i \sin\varphi$$

$$\text{Then, } z^k = \cos(k\varphi) + i \sin(k\varphi)$$

- * Roots of Polynomials

Fundamental Theorem of Algebra - a polynomial of degree n has exactly n possibly multiple complex roots

$$P_n(x) = c_1 (x-x_1)^{m_1} (x-x_2)^{m_2} \dots (x-x_k)^{m_k}$$

x_1, \dots, x_k - distinct roots with multiplicities m_1, \dots, m_k

$$m_1 + m_2 + \dots + m_k = n$$

Remark

Given the numbers $c_1, c_2, \dots, c_{n+1} \in \mathbb{R}$, consider an (n+1) companion matrix

$$A_n = \frac{1}{c_1} \begin{bmatrix} 0 & 0 & \dots & 0 & -c_{n+1} \\ c_1 & 0 & \dots & 0 & -c_n \\ 0 & c_1 & \dots & 0 & -c_{n-1} \\ \vdots & & & \vdots & \vdots \\ 0 & 0 & \dots & c_1 & c_2 \end{bmatrix}$$

It can be shown that

$$p_n(z) = \det(A - zI)$$

and the roots of $p_n(z)$ are given by the eigenvalues of the companion matrix A . Solving the eigenvalue problem for the companion matrix A the

- general way of finding the roots of a polynomial (MATLAB's function roots), because
- all roots are found in one computation (both real and complex)
 - it is a numerically stable and efficient procedure (no issue of the initial guess, etc.)

* Representation of polynomials : ~~over~~ power series (coefficients) vs. factorized (roots + c_i)

* Differentiation and integration of polynomials is trivial

$$P_n(x) = n c_1 x^{n-1} + (n-1) c_2 x^{n-2} + \dots + c_n$$

$$\int P_n(x) dx = \frac{c_1}{n+1} x^{n+1} + \frac{c_2}{n} x^n + \dots + \frac{c_n}{2} x^2 + c_{n+1} x + c_{n+2}$$

* Multiplication of polynomials $P_n(x) \cdot P_m(x) = P_{n+m}(x)$

$$\left(\sum_{i=1}^{n+1} a_i x^{n+1-i} \right) \left(\sum_{j=1}^{m+1} b_j x^{m+1-j} \right) = \sum_{l=1}^{n+m+1} c_l x^{n+m+1-l}$$

c_l - complicated functions of $\{a_i\}$ and $\{b_j\}$

* Division of polynomials leads to rational functions (not polynomials anymore)

$$R(x) = \frac{P_n(x)}{P_m(x)}$$

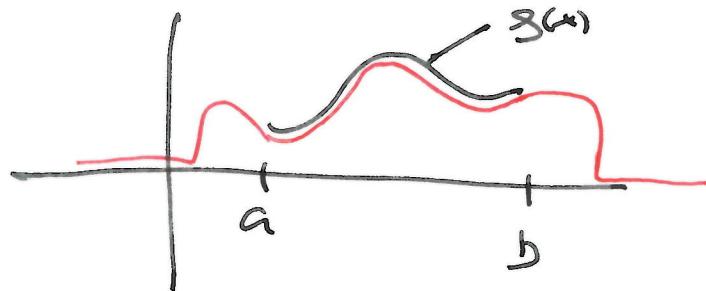
* Weierstrass Approximation Theorem (1885)

Let f be a continuous function on $[a, b]$ and let $\epsilon > 0$ be arbitrary. Then, there exists a polynomial p_n such that

$$\max_{a \leq x \leq b} |p_n(x) - f(x)| < \epsilon$$

Proof (outline) (Original proof given by Wiener-Hopf)

- extend \tilde{g} to a continuous function $\tilde{\tilde{g}}$ with compact support defined on the real line



- consider an initial-value problem for the diffusion equation $\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}$ on the real line $x \in \mathbb{R}$; as the initial data at $t=0$ take $\tilde{\tilde{g}}$, i.e., $u(0, x) = \tilde{\tilde{g}}(x), x \in \mathbb{R}$
- The solution is given by convolving the convolution of $\tilde{\tilde{g}}$ with the Gaussian (heat) Kernel $G(t, x) = \frac{1}{\sqrt{4\pi t}} e^{-\frac{x^2}{4t}}$
$$u(t, x) = \int_{-\infty}^{\infty} \tilde{\tilde{g}}(x') G(t, x - x') dx'$$

Note that $u(t, x) \rightarrow \tilde{\tilde{g}}(x)$ uniformly in x as $t \rightarrow 0$. Thus, $u(t, x)$ can be arbitrarily close to $\tilde{g}(x)$ for $x \in [a, b]$. Due to smoothness of the kernel $G(t, x)$, ~~and~~ $\forall t > 0$ $u(t, x)$ is analytic in x . Thus, $u(t, x)$ has a uniformly convergent Taylor Series which can be truncated at a suitable order to give a polynomial such that

$$\max_{a \leq x \leq b} |P_n(x) - \tilde{g}(x)| < \epsilon$$

Remarks

- * central result of approximation theory
- * the theorem is not constructive; it's not even known what the degree of the polynomial should be

2.1 Interpolating Polynomials

Given $(N+1)$ points $\{(x_i, y_i)\}_{i=0}^N$, find a ~~deg~~ degree N polynomial passing through all these points.

$$g(x) = \sum_{k=0}^N a_k x^k \text{ has } (N+1) \text{ unknown coefficients}$$

which can be determined using the following conditions

$$\left\{ \begin{array}{l} \sum_{k=0}^N a_k x_0^k = y_0 \\ \sum_{k=0}^N a_k x_1^k = y_1 \\ \vdots \\ \sum_{k=0}^N a_k x_N^k = y_N \end{array} \right.$$

$$\left[\begin{array}{cccc|c} 1 & x_0 & \dots & x_0^n \\ 1 & x_1 & \dots & x_1^n \\ \vdots & & & \vdots \\ 1 & x_N & \dots & x_N^n \end{array} \right] \left[\begin{array}{c} a_0 \\ a_1 \\ \vdots \\ a_N \end{array} \right] = \left[\begin{array}{c} g_0 \\ \vdots \\ g_N \end{array} \right]$$

Vandermonde
matrix ✓

Vandermonde determinant: $\text{Det}(V) = \prod_{i=0}^N \prod_{j=i+1}^N (x_i - x_j)$

Thus, if N distinct interpolation points $x_i \neq x_j$, $i \neq j$,
 $\text{Det}(V) \neq 0$ and the Vandermonde matrix is
nonsingular \Rightarrow unique solutions exist

Theorem

There exists a unique interpolating polynomial $P_n(x)$
 iff the data points x_0, x_1, \dots, x_N are distinct.

However, the Vandermonde matrix is extremely ~~the~~
 ill-conditioned, to the point that it is very
 difficult to use in practice.

Reason — The basis functions are monomials x^k , $k=0, \dots, N$
 which for large k look alike; in analogy with
 vectors in \mathbb{R}^n , they are almost collinear (linearly
 dependent).