Interpolants, Projections & Aliasing

(Trefethen, Chapter 4)

Suppose $f : [-1,1] \to \mathbb{R}$ is Lipschitz continuous. It then has a Chebyshev series with coefficients a_k, $k = 0, 1, \ldots$

$$f(x) = \sum_{k=0}^{\infty} a_k T_k(x)$$

There are two distinct ways of representing $f(x)$ approximately as a polynomial in P_n.

* interpolation on the Chebyshev points

$$p_n(x) = \sum_{k=0}^{n} c_k T_k(x)$$

where $\quad p_n(x_j) = f(x_j) \quad , \; x_j = \cos\left(\frac{j\pi}{n}\right), \; 0 \le j \le n$

* truncation of the Chebyshev series

$$f_n(x) = \sum_{k=0}^{n} a_k T_k(x)$$

<u>Question</u> : what is the relation between:

- the coefficients a_k and c_k, $k = 0, \ldots, n$
- the corresponding functions $f_n(x)$ and $p_n(x)$?

Aliasing – consider a Chebyshev grid with $(n+1)$ points; when evaluated on this grid, any Chebyshev polynomial T_N (no matter how big N is) is indistinguishable from a single Chebyshev polynomial T_m for some m $0 \leq m \leq n$.

Ex – aliasing_01.m

Theorem (Aliasing of Chebyshev Polynomials)
For any $n \geq 1$ and $0 \leq m \leq n$, the following Chebyshev polynomials take the same values on the $(n+1)$-point Chebyshev grid

$$T_m, \; T_{2n-m}, \; T_{2n+m}, \; T_{4n-m}, \; T_{4n+m}, \; T_{6n-m}, \ldots$$

Equivalently, for any $k \geq 0$, T_k takes the same value on the grid as T_m with

$$m = |(k+n-1) \pmod{2n} - (n-1)|$$

a number in the range $0 \leq m \leq n$.

Proof (the first assertion only; for the proof of the second assertion, see Trefethen's book)

Chebyshev polynomials are related to the complex monomials on the unit circle as

$$T_m(x) = \frac{1}{2}(z^m + z^{-m}) = \cos(m\theta)$$

$z = e^{i\theta}$ $\qquad \theta = \cos^{-1}(x)$

The Chebyshev points are related to ~~the~~ $(2n)$th roots of unity as

Thus,

$$x_m = \frac{1}{2}(z_m - z_m^{-1})$$

The first assertion is equivalent to the statement that the following functions

$$z^m + z^{-m}, \quad z^{2n-m} + z^{m-2n}, \quad z^{2n+m} + z^{-2n-m}, \quad \dots$$

take the same values at the $(2n)$th ~~roots~~ roots of unity. Modulo $2n$, all the exponents are equal to $\pm m$. The assertion then follows from the property of $(2n)$th roots of unity

$$z^{2n} = 1 \quad \Rightarrow \quad (z^{2n})^l = z^{2ln} = 1 \quad \forall\, l \in \mathbb{N}$$

$\blacksquare$

<u>Corollary</u>

The coefficients $\{c_k\}$ of the Chebyshev interpolant are related to the Chebyshev expansion coefficients $\{a_k\}$ as follows

$$c_0 = a_0 + a_{2n} + a_{4n} + \dots$$

$$\vdots$$

$$c_n = a_n + a_{3n} + a_{5n} + \dots$$

or, for $1 \leq k \leq n-1$

$$c_k = a_k + (a_{k+2n} + a_{k+4n} + \dots) + (a_{-k+2n} + a_{-k+4n} + \dots)$$