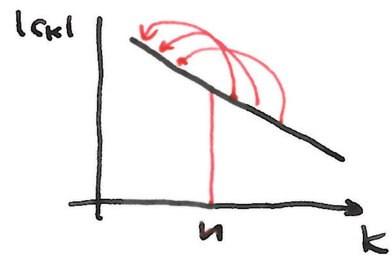
Conclusion – on the $(n+1)$-point grid, any function f is indistinguishable from a degree n polynomial. In particular, the Chebyshev series of the polynomial interpolant $p_n(x)$ to $f(x)$ is obtained by reassigning all the Chebyshev coefficients in the infinite series for $f(x)$ to their aliases of degrees $0, \dots, n$

Thus:

$$f(x) - f_n(x) = \sum_{k=n+1}^{\infty} a_k T_k(x)$$

$$f(x) - p_n(x) = \sum_{k=n+1}^{\infty} a_k \left[T_k(x) - T_m(x) \right]$$

$$m = \left| (k+n-1) \pmod{2n} - (n-1) \right|$$



Remark

As will be shown by the following example (aliasing_02.m), the truncation f_n provides a slightly better representation of f than the Chebyshev interpolant p_n, but the difference is very small and they behave similarly. However, the interpolants p_n are much easier and faster to evaluate (via a suitable adaptation of the barycentric formula), and they are the basis of CHEBFUN.

Aliasing is a well-known phenomenon in <u>signal processing</u> when signals ~~[illegible]~~ containing certain frequencies are undersampled at lower frequencies.

<u>Convergence of Chebyshev Expansions for Differentiable Functions</u> (Trefethen, Chapter 7)

Recall that for $f:[-1,1]\to\infty$, $\|f\|_\infty = \sup_{-1\le x\le 1}|f(x)|$

<u>Question</u>: What is the relation between $\|f-p_n\|_\infty$, and $\|f-f_n\|_\infty$ and the properties of the function f?
In particular, how do these errors decrease as $n\to\infty$?

Suppose f is ν times differentiable, possibly with jumps in the $(\nu+1)$-st derivative

<u>Ex</u>: cheb_convergence_01.m

$\mathrm{Err} \sim C n^{-\alpha}$

$\log \mathrm{Err} \sim \log C - \alpha \log n$

<u>Total variation</u>: $\|f\|_{L^1} = \int_{-1}^{1} |f(x)|\,dx$

$$(\text{or } \|f^{(\nu+1)}\|_{L^1} = \int_{-1}^{1} |f^{(\nu+1)}(x)|\,dx$$

The $(\nu+1)$th derivative of the function $f(x)$ may be discontinuous, but may also have <u>finite total variation</u>

<u>Ex</u>. $\|(|x|)'\|_{L^1} = \int_{-1}^{1} |(|x|)'| = 2$

Theorem (Chebyshev coefficients of differentiable functions)

For an integer $\nu \geq 0$, let f and its derivatives $f^{(\nu-1)}$ be absolutely continuous on $[-1,1]$ and suppose the ν-th derivative $f^{(\nu)}$ is of bounded variation V. Then, for $k \geq \nu+1$, the Chebyshev coefficients of f satisfy

$$|a_k| \leq \frac{2V}{\pi (k-\nu)^{\nu+1}}$$

Proof - See Trefethen's book

Theorem (Convergence for differentiable functions)

If f satisfies the conditions of the previous theorem, then for any $n > \nu$, its Chebyshev projections satisfy

$$\| f - f_n \| \leq \frac{2V}{\pi \nu (n-\nu)^{\nu}}$$

and its Chebyshev interpolants satisfy

$$\| f - p_n \| \leq \frac{4V}{\pi \nu (n-\nu)^{\nu}}$$

Proof

$$\| f - f_n \| \leq \sum_{k=n+1}^{\infty} |a_k| \leq \left| \begin{matrix}\text{from the} \\ \text{previous} \\ \text{theorem}\end{matrix} \right| \leq \frac{2V}{\pi} \sum_{k=n+1}^{\infty} (k-\nu)^{-\nu-1}$$

which can be bounded by $\int_n^{\infty} (s-\nu)^{-\nu-1} \, ds = \frac{1}{\nu (n-\nu)^{\nu}}$

The bound on $\| f - p_n \|$ is obtained similarly, but we need to account for aliasing.

Thus,

νth derivative of bounded variation $\Longrightarrow$ the error of Chebyshev approximation / interpolation decreases at an <u>algebraic</u> rate $O(n^{-\nu})$

The error bound for interpolation is worse by merely a factor of 2.

<u>Convergence of Chebyshev Expansions for Analytic Functions (Trefethen, Chapter 8)</u>

Suppose now that f is analytic on $[-1,1]$ (i.e., $f \in C^\infty([-1,1])$ and f admits a convergent Taylor series about any $s \in [-1,1]$)

<u>Bernstein ellipses</u> – the largest ellipse E_ρ in the complex plane with foci at ± 1 in which the ~~extension~~ complex extension $f(z)$ of the function is <u>analytic and bounded</u>

$$\rho = \begin{bmatrix}\text{length of} \\ \text{semiminor} \\ \text{axis}\end{bmatrix} + \begin{bmatrix}\text{length of} \\ \text{semimajor} \\ \text{axis}\end{bmatrix}$$

Theorem (Chebyshev coefficients of analytic functions)

Let a function f be analytically continuable to the open Bernstein ellipse E_ρ, where it satisfies $|f(z)| \le M$ for some $M>0$. Then, its Chebyshev coefficients satisfy $|a_0| \le M$ and

$$|a_k| \le 2M\rho^{-k}, \quad k \ge 1$$

Theorem (Convergence for Analytic functions)

If f has the properties mentioned in the theorem above, then for each $n \geq 0$ its Chebyshev projections satisfy

$$\|f - f_n\| \le \frac{2M\rho^{-n}}{\rho - 1}$$

and its Chebyshev interpolants satisfy

$$\|f - p_n\| \le \frac{4M\rho^{-n}}{\rho - 1}$$

Proof:- see chapter 8 of Trefethen's book

Remark - convergence is faster for functions $f(x)$ with complex-plane singularities further away from the real axis (larger Bernstein ellipses with larger ρ); in particular, for entire functions $\rho \to \infty$

Ex - bernstein.m, cheb-convergence_02.m

From the data: $\log \mathrm{Err} \sim C' - 2n = \log(C\,10^{-2n})$

$$\mathrm{Err} \sim C\,10^{-2n}$$

10 can be replaced by any number by taking $\log_a$ with a suitable base a.