MATH 3Q03 - TEST #1 - SOLUTIONS

① The value of the variable a will be exactly 10.0. The addition of 10^{-20} has no effect due to round-off errors in the finite-precision arithmetic (10^{-20} is smaller than eps(10)).

2

② For problems of the type $f(x)=0$, the linear convergence means that the approximation error $|e_n| = |x_n - R|$ at a given iteration, where R is the root and x_n the corresponding approximation, is a linear function of the error at the previous iteration, i.e.,

$$|e_n| = C_1 |e_{n-1}|.$$

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The quadratic convergence means that the error at the given iteration is a quadratic function of the error at the previous iteration, i.e.,

$$|e_n| = C_2 |e_{n-1}|^2$$

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Examples of root-finding methods with

- linear convergence: bisection, fixed-point iteration
- quadratic convergence: Newton's method

③

* transform the original problem $f(x)=0$ to a fixed-point form, $g(x)=x$, e.g., by defining
$g(x) = f(x) + x$ ①

* set up fixed-point iterations

$$x_{n+1} = g(x_n), \quad n=1,2,\dots$$

with some initial guess x_1. Then $\lim_{n\to\infty} x_n = R$ (root).

* convergence analysis ①

$$x_{n+1} = g(x_n) \Rightarrow R - x_{n+1} = R - g(x_n) = g(R) - g(x_n)$$
$$= \frac{g(R)-g(x_n)}{R-x_n}(R-x_n)$$

Next, use the <u>mean-value theorem</u>

$$R - x_{n+1} = g'(\xi_n)(R-x_n), \quad \text{where } \xi_n \in [x_n, R]$$

so that, denoting $e_n = R - x_n$, we obtain

$$|e_{n+1}| = |g'(\xi_n)|\,|e_n| \rightarrow \text{\underline{linear convergence}}$$

Thus, iterations will converge if $|g'(\xi)|<1$ for ξ in some interval

In fact, this condition can be weakened to Lipschitz-continuity with the Lipschitz constant $L<1$ (known as <u>contractivity</u>)

(4) In the bisection method the error at the n-th iteration can be upper-bounded as

$$|R - x_n| \leq \frac{|b-a|}{2^n} = \frac{1}{2^n}$$ 1

We need $\frac{1}{2^n} \leq 10^{-3}$

$$-n \log_{10} 2 \leq -3 \log_{10} 10 = -3$$

$$n \geq \frac{3}{\log_{10} 2} \cong 9.9658$$ 1

We need at most 10 iterations.

(5)

a) $\mathrm{Det}\begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix} = 0 \Rightarrow$ the system is singular and has <u>infinitely many</u> solutions in the form

1

$x = \alpha v_1$, $\alpha \in \mathbb{R}$, where $v_1 = [-2 \;\; 1]^2$ is the eigenvector associated with the zero eigenvalue.

b) $\mathrm{Det}\begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} = -3 \Rightarrow$ the system is nonsingular and admits a unique solution

1