MATH 3Q03 TEST #2 - SOLUTIONS

(1) Derivation of Newton's method:

* Expand $\bar{F}(\bar{x}) = 0$ in a Taylor series around $\bar{x}_k$

$$\bar{F}(\bar{x}) = \bar{F}(\bar{x}_k) + \nabla \bar{F}(\bar{x}_k)(\bar{x} - \bar{x}_k) + O(\|\bar{x} - \bar{x}_k\|^2)$$

* Evaluate the expansion at $\bar{x} = \bar{x}_{k+1}$, set $\bar{F}(\bar{x}_{k+1}) = 0$ and neglect the quadratic terms

(1)

$$\nabla \bar{F}(\bar{x}_k) \cdot \underbrace{(\bar{x}_{k+1} - \bar{x}_k)}_{\text{Newton step}} = -\bar{F}(\bar{x}_k)$$

* rearrange

$$(*) \quad \bar{x}_{k+1} = \bar{x}_k - [\nabla \bar{F}(\bar{x}_k)]^{-1} \bar{F}(x_k), \quad k = 1, 2,$$

At every iteration of Newton's method we need to:

- evaluate the function at $\bar{x}_k$; $\bar{F}(x_k)$ $(\frac{1}{2})$
- evaluate and invert the Jacobian at $\bar{x}_k$: $[\nabla F(\bar{x}_k)]^{-1}$
- update the approximation with formula $(*)$

For Newton's method to apply, the function $\bar{F}(\bar{x})$:

- must be differentiable
- and its Jacobian nonsingular $(\frac{1}{2})$

in the neighbourhood of the root $\bar{x}^*$.

Furthermore, to ensure convergence of iterations, we also need $\rho(\nabla \bar{F}(\bar{x})) < 1$, where $\rho(\cdot)$ is the spectral radius of the matrix, in the neighbourhood of the root.

(2) For the Weierstrass Approximation Theorem to apply:

- the function $f(x)$ must be at least ~~continuous~~ continuous,
- its domain of definition $[a,b]$ must be bounded (i.e., $a > -\infty$, $b < \infty$).

(a) $x \in \mathbb{R} \Rightarrow$ unbounded domain
Theorem will not apply

$\frac{1}{2}$

(b) $f(x)$ is continuous
domain $[0, 2\pi]$ bounded $\Bigg\} \Rightarrow$ theorem will apply

$\frac{1}{2}$

(c) $f(x)$ discontinuous at $x = 0 \Rightarrow$ theorem will not apply

$\frac{1}{2}$

(d) $f(x)$ is continuous
domain $[-1, 1]$ bounded $\Bigg\} \Rightarrow$ theorem will apply

$\frac{1}{2}$

③ Define: $x_0=-1,\ y_0=0,\ x_1=0,\ y_1=-1,\ x_2=1,\ y_2=2$

The interpolating polynomial is then given by

$$p_2(x) = \underbrace{\phi_0 y_0}_{0} + \phi_1 y_1 + \phi_2 y_2 \quad ①$$

where ϕ_0, ϕ_1, ϕ_2 are the Lagrange functions with the property $\phi_i(x_j) = \delta_{ij}$, $i,j=0,1,2$. Because $y_0=0$, ϕ_0 is irrelevant.

$$\phi_1(x) = \frac{(x-x_0)(x-x_2)}{(x_1-x_0)(x_1-x_2)} = \frac{(x+1)(x-1)}{(0+1)(0-1)} = 1-x^2 \quad ①$$

$$\phi_1(x_1) = \phi_1(0) = 1, \quad \phi_1(\pm 1) = 0$$

$$\phi_2(x) = \frac{(x-x_0)(x-x_1)}{(x_2-x_0)(x_2-x_1)} = \frac{(x+1)x}{2\cdot 1} = \frac{1}{2}x(x+1)$$

$$\phi_2(x_0) = \phi_2(-1) = 0; \quad \phi_2(x_1) = \phi_2(0) = 0 \quad ①$$

$$\phi_2(x_2) = \phi_2(1) = \frac{1}{2}\cdot 1\cdot 2 = 1$$

Then, finally,

$$p_2(x) = (-1)(1-x^2) + 2\,\frac{1}{2}x(x+1)$$

$$= x^2-1+x^2+x = \underline{2x^2+x-1} \quad ①$$

④ The interpolation error is described as

$$E(x) = f(x) - p_n(x) = \frac{f^{(n+1)}(\xi)}{(n+1)!} \prod_{i=0}^{n} (x - x_i), \quad \xi \in [-1,1]$$

Define $M_n = \max_{x \in [-1,1]} |f^{(n)}(x)|$

For the two functions the behavior of M_n will be quite different:

ⓐ for ~~e~~ $f(x) = e^x$, M_n will remain bounded for all n and interpolation will converge 1

ⓑ for $f(x) = e^{|x|}$, due to non-differentiability at $x=0$, M_n will be unbounded already for $n \geq 1$ and interpolation will diverge. 1