MATH 3Q03: Finite Difference Methods for Differential Equations

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PART II
Finite Difference Methods for Differential Equations

Agenda

- Solving a TWO-POINT BOUNDARY VALUE PROBLEM with DIRICHLET BOUNDARY CONDITIONS :

$$\frac{d^2y}{dx^2} = g \qquad \text{for } x \in (0, 2\pi)$$
$$y(0) = y(2\pi) = 0$$

- Finite-difference approximation:
  - Second-order center difference formula for the interior nodes:

$$\frac{y_{j+1} - 2y_j + y_{j-1}}{h^2} = g_j \text{ for } j = 1, \dots, N$$

    where $h = \frac{2\pi}{N+1}$ and $x_j = jh$
  - Endpoint nodes:

$$y_0 = 0 \quad \Longrightarrow y_2 - 2y_1 = h^2 g_1$$
$$y_{N+1} = 0 \Longrightarrow -2y_N + y_{N-1} = h^2 g_N$$

  - Tridiagonal algebraic system — solved very efficiently with the THOMAS ALGORITHM (a version of the Gaussian elimination)

- Solving a TWO-POINT BOUNDARY VALUE PROBLEM with NEUMANN BOUNDARY CONDITIONS :

$$\frac{d^2y}{dx^2} = g \qquad \text{for } x \in (0, 2\pi)$$

$$\frac{dy}{dx}(0) = \frac{dy}{dx}(2\pi) = 0$$

- Finite-difference approximation:
  - Second-order center difference formula for the interior nodes:

$$\frac{y_{j+1} - 2y_j + y_{j-1}}{h^2} = g_j \text{ for } j = 1, \dots, N$$

  - First-order Forward/Backward Difference formulae to re–express endpoint values:

$$\frac{y_1 - y_0}{h} = 0 \implies y_0 = y_1$$

$$\frac{y_{N+1} - y_N}{h} = 0 \implies y_{N+1} = y_N$$

    First-order only — DEGRADED ACCURACY!
  - Tridiagonal algebraic system — Is there any problem? Where?

- In order to retain the SECOND-ORDER ACCURACY in the approximation of the Neumann problem need to use higher-order formulae at endpoints, e.g.

$$y'_0 = \frac{-y_2 + 4y_1 - 3y_0}{2h} = 0 \implies y_0 = \frac{1}{3}(-y_2 + 4y_1)$$

- The first row thus becomes

$$\frac{2}{3}y_2 - \frac{2}{3}y_1 = h^2 g_1$$

SECOND–ORDER ACCURACY RECOVERED!

- COMPACT STENCILS — stencils based on three grid points (in every direction) only: $\{x_{j+1}, x_j, x_{j-1}\}$ at the $j - th$ node
- Is is possible to obtain higher (then second) order of accuracy on compact stencils? — YES!
- Consider the central difference approximation to the equation $\frac{d^2y}{dy^2} = g$

$$\frac{y_{j+1} - 2y_j + y_{j-1}}{h^2} - \frac{h^2}{12} y_j^{(iv)} + \mathcal{O}(h^4) = g_j$$

- Re-express the error term $\frac{h^2}{12} y_j^{(iv)}$ using the equation in question:

$$\frac{h^2}{12} y_j^{(iv)} = \frac{h^2}{12} g_j'' = \frac{h^2}{12} \left[\frac{g_{j+1} - 2g_j + g_{j-1}}{h^2} - \frac{h^2}{12} g_j^{(iv)} + \mathcal{O}(h^4) \right]$$

- Inserting into the original finite-difference equation:

$$\frac{y_{j+1} - 2y_j + y_{j-1}}{h^2} = g_j + \frac{g_{j+1} - 2g_j + g_{j-1}}{12} + \mathcal{O}(h^4)$$

- Slight modification of the RHS $\implies$ FOURTH-ORDER ACCURACY!!!

- COMPACT FINITE DIFFERENCE SCHEMES —
  - ADVANTAGES:
    - Increased accuracy on compact stencils
  - DRAWBACKS:
    - need to be tailored to the specific equation solved
    - can get fairly complicated for more complex equations

- Consider the following CAUCHY PROBLEM :

$$\frac{dy}{dt} = f(y,t) \text{ with } y(t_0) = y_0$$

The independent variable t is usually referred to as TIME .

- Equations with higher-order derivatives can be reduced to systems of first-order equations

- Generalizations to systems of ODEs straightforward

- When the RHS function does not depend on y, i.e., $f(y,t) = f(t)$, solution obtained via a QUADRATURE

- Assume uniform time-steps (h is constant)

- ACCURACY — unlike in the Boundary Value Problems, there is no terminal condition and approximation errors may accumulate in time; consequently, a relevant characterization of accuracy is provided by the GLOBAL ERROR

$$\text{(global error)} = \text{(local error)} \times (\#\ \text{of time steps}),$$

rather than the LOCAL ERROR .

- STABILITY — unlike in the Boundary Value Problems, where boundedness of the solution at final time is enforced via a suitable terminal condition , in Initial Value Problems there is a priori no guarantee that the solution will remain bounded.

Model Problem

- STABILITY of various numerical schemes is usually analyzed by applying these schemes to the following LINEAR MODEL :

$$\frac{dy}{dt} = \lambda y = (\lambda_r + i\lambda_i)y \text{ with } y(t_0) = y_0,$$

which is stable when $\lambda_r <= 0$.

- EXACT SOLUTION:

$$y(t) = y_0 e^{\lambda t} = \left(1 + \lambda h + \frac{\lambda^2 h^2}{2} + \frac{\lambda^3 h^3}{6} + \dots\right) y_0$$

Euler Explicit Scheme (I)

- Consider a Taylor series expansion

$$y(t_{n+1}) = y(t_n) + hy'(t_n) + \frac{h^2}{2}y''(t_n) + \dots$$

Using the ODE we obtain

$$y' = \frac{dy}{dt} \quad = f$$
$$y'' = \frac{dy'}{dt} = \frac{df}{dt} = f_t + ff_y$$

- Neglecting terms proportional to second and higher powers of h yields the Explicit Euler Method

$$y_{n+1} = y_n + hf(y_n, t_n)$$

- Retaining higher–order terms is inconvenient, as it requires differentiation of f and does not lead to schemes with desirable stability properties.

Euler Explicit Scheme (II)

- LOCAL ERROR analysis:

$$y_{n+1} = (1 + \lambda h)\, y_n + [\mathcal{O}(h^2)]$$

- GLOBAL ERROR analysis:

$$(\text{global error}) = Ch^2 \cdot N = Ch^2 \cdot \frac{T}{h} = C' h$$

Thus, the scheme is

- locally second-order accurate
- globally (over the interval $[t_0, t_0 + Nh]$) first-order accurate

Euler Explicit Scheme (III)

- Stability (for the model problem)

$$y_{n+1} = y_n + \lambda h y_n = (1 + \lambda h) y_n$$

- Thus, the solution after n time steps

$$y_n = (1 + \lambda h)^n y_0 \triangleq \sigma^n y_0 \quad \Longrightarrow \quad \sigma = 1 + \lambda h$$

- For large n, the numerical solution remains stable iff

$$|\sigma| \leq 1 \quad \Longrightarrow \quad (1 + \lambda_r h)^2 + (\lambda_i h)^2 \leq 1$$

  - CONDITIONALLY STABLE for real λ
  - UNSTABLE for imaginary λ

Euler Implicit Scheme (I)

- IMPLICIT SCHEMES — based on approximation of the RHS that involve $f(y_{n+1}, t)$, where y_{n+1} is the unknown to be determined
- IMPLICIT EULER SCHEME — obtained by neglecting second and higher-order terms in the expansion:

$$y(t_n) = y(t_{n+1}) - hy'(t_{n+1}) + \frac{h^2}{2} y''(t_{n+1}) - \dots$$

- Upon substitution $\left.\frac{dy}{dt}\right|_{t_{n+1}} = f(y_{n+1}, t_{n+1})$ we obtain

$$y_{n+1} = y_n + hf(y_{n+1}, t_{n+1})$$

- The scheme is
  - locally SECOND–ORDER accurate
  - globally (over the interval $[t_0, t_0 + Nh]$) FIRST–ORDER accurate

Euler Implicit Scheme (II)

- Stability (for the model problem):

$$y_{n+1} = y_n + \lambda h y_{n+1} \quad \Longrightarrow \quad y_{n+1} = (1-\lambda h)^{-1} y_n$$

$$y_{n+1} = \left(\frac{1}{1-\lambda h}\right)^n y_0 \triangleq \sigma^n y_0 \quad \Longrightarrow \quad \sigma = \frac{1}{1-\lambda h}$$

$$|\sigma| \le 1 \quad \Longrightarrow \quad (1-\lambda_r h)^2 + (\lambda_i h)^2 \ge 1$$

- Implicit Euler scheme is thus stable for
  - all stable model problems
  - most unstable model problems
- REMARK: When solving systems of ODEs of the form $\mathbf{y} = \mathcal{A}(t)\mathbf{y}$, each implicit step requires solution of an algebraic system: $\mathbf{y}_{n+1} = (I - h\mathcal{A})^{-1}\mathbf{y}_n$
- Implicit schemes are generally hard to implement for nonlinear problems