HOMEWORK #2

Due: February 16 (Thursday) by midnight

Instructions:

- The assignment consists of *three* questions, worth 4, 3 and 3 points.
- Submit your assignment *electronically* to the Email address `math3q03@math.mcmaster.ca`; hardcopy submissions will not be accepted.
- It is obligatory to use the MATLAB template file available at `http://www.math.mcmaster.ca/bprotas/MATH3Q03/template.m` (see also the link in the "Computer Programs" section of the course website); submissions non compliant with this template will not be accepted. **Please make sure to use the latest version of this file (which has been updated).**
- All plots should have suitable axis labels and legends.
- Make sure to enter your name and student I.D. number in the appropriate section of the template.
- Late submissions and submissions which do not comply with these guidelines will not be accepted.

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1. The purpose of this problem is to illustrate how well the matrix norm $||\mathbf{A}||_p$ describes the action of a matrix $\mathbf{A}$ on some vector $\mathbf{x}$. Suppose that

$$\mathbf{A} = \begin{bmatrix} 1 & 2 & 2 & 0 \\ 2 & 3 & -1 & 0 \\ 1 & -4 & 2 & -3 \\ -2 & 0 & 1 & 1 \end{bmatrix} \tag{1}$$

and generate $M = 10^4$ 4×1 vectors $\mathbf{x}_i$, $i = 1, \dots, M$, whose components are random numbers distributed uniformly over the interval $[0, 1]$ (each of these vectors can be created in MATLAB using the function `rand(...)`). Then, for each of these vectors compute the quantities

$$q_{p,i} = \frac{||\mathbf{A}\mathbf{x}_i||_p / ||\mathbf{x}_i||_p}{||\mathbf{A}||_p}, \quad i = 1, \dots, M,$$

where $p = 1, 2, \infty$. Note that $q_{1,i}, q_{2,i}, q_{\infty,i} \in [0, 1]$, $i = 1, \dots, M$. Finally, compute the *probability density functions (PDFs)* of the random variables $q_{1,i}$, $q_{2,i}$ and $q_{\infty,i}$, $i = 1, \dots, M$, and plot these three PDFs in figure 1 using different line styles and colors. In each case the PDF can be obtained by dividing the interval $[0, 1]$ into 30 equal "bins" and then assigning the random variables to these bins. When evaluating vector and matrix norms write your own function(s) and do not use the MATLAB function `norm`.
(4 points)

2. You are given an algebraic system $\mathbf{C}\mathbf{x} = \mathbf{b}$, where

$$\mathbf{C} = \begin{bmatrix} 6 & 1 & 0 & 3 \\ 5 & 8 & 4 & 0 \\ 2 & 2 & 7 & 1 \\ 0 & 1 & 2 & 4 \end{bmatrix} \tag{2}$$

and right–hand side (RHS) vector is $\mathbf{b} = [1 \;\; 1 \;\; 1 \;\; 1]^T$.

(a) Using the MATLAB function `lu` perform the LU decomposition of the matrix $\mathbf{C}$; print out the resulting matrices $\mathbf{L}$ and $\mathbf{U}$ (note that, due to the specific structure of the matrix $\mathbf{C}$, the matrix $\mathbf{L}$ returned by the function `lu` should indeed be lower triangular).

(b) Write your own two functions `Lbck` and `Ubck` that solve an algebraic system with, respectively, lower and upper triangular matrix using back and forward substitution; use these two functions and the matrices $\mathbf{L}$ and $\mathbf{U}$ to solve the system $\mathbf{Cx} = \mathbf{b}$, then print the solution $\mathbf{x}$ and store it in the variable `Answer1`.

(c) Solve the system $\mathbf{Cx} = \mathbf{b}$ directly using the "backslash" operator "\", then print the solution $\mathbf{x}$ and store it in the variable `Answer2`.

(3 points)

3. Consider an accelerated version of the Gauß-Seidel iterative method, known as the *Successive Over-Relaxation (SOR)*. Given a number $\omega \in (0, 2)$ and the algebraic system

$$\mathbf{Ax} = (\mathbf{L} + \mathbf{D} + \mathbf{U})\mathbf{x} = \mathbf{b},$$

where $\mathbf{L}$, $\mathbf{D}$ and $\mathbf{U}$ are, respectively, the strictly lower-triangular, diagonal and upper-triangular matrices, we can rewrite it in the fixed-point form as

$$(\omega\mathbf{D} + \omega\mathbf{L})\mathbf{x} = -\omega\mathbf{U}\mathbf{x} + \omega\mathbf{b}. \tag{3}$$

After simple manipulations, relation (3) can be transformed to the following iterative algorithm

$$\mathbf{x}^{(n+1)} = (\mathbf{D} + \omega\mathbf{L})^{-1}\left\{-\left[\omega\mathbf{U} + (\omega - 1)\mathbf{D}\right]\mathbf{x}^{(n)} + \omega\mathbf{b}\right\}, \quad n = 1, 2, \dots, \tag{4}$$

where n denotes the iteration count. Use approach (4) to solve the algebraic problem (with the "poisson" matrix) defined in the function `compare_err.m` (posted as a part of "Iterative Solvers (I)" on the course website). In the computations use $N = 10$ (so that the system of linear equations has $N^2 = 100$ unknowns) and the tolerance of 10^{-6}. Obtain solutions for the following values of the parameter $\omega = 0.5, 1.0, 1.5$ and also solve the problem using the standard Gauß-Seidel method. In each case plot the solution error as a function of iterations, as done by the function `compare_err.m`. This plot should appear as `figure(2)`.
(3 points)