HOMEWORK #3

Due: March 9 (Thursday) by midnight

Instructions:

- The assignment consists of *three* questions, worth 3, 4 and 3 points.
- Submit your assignment *electronically* to the Email address `math3q03@math.mcmaster.ca`; hardcopy submissions will not be accepted.
- It is obligatory to use the MATLAB template file available at `http://www.math.mcmaster.ca/bprotas/MATH3Q03/template.m` (see also the link in the "Computer Programs" section of the course website); submissions non compliant with this template will not be accepted.
- All plots should have suitable axis labels and legends.
- Make sure to enter your name and student I.D. number in the appropriate section of the template.
- Late submissions and submissions which do not comply with these guidelines will not be accepted.

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1. Consider the problem of finding roots of the function $\mathbf{F} : \mathbb{R}^2 \to \mathbb{R}^2$, where

$$\mathbf{F}\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} x_1 x_2 - 1 \\ x_2^2 - x_1 - 2 \end{bmatrix}.$$

When the Jacobian of $\mathbf{F}$ is too difficult or too costly to evaluate, Broyden's method is a viable alternative to Newton's method. Starting from initial guesses $\mathbf{x}_0$ and $\mathbf{D}_0$, the iterations of Broyden's approach are defined as follows (see Section 8.2 in Grasselli & Pelinovsky)

$$\mathbf{D}_k(\mathbf{x}_{k+1} - \mathbf{x}_k) = -\mathbf{F}(\mathbf{x}_k), \quad k = 0, 1, \dots$$

$$\mathbf{D}_{k+1} = \mathbf{D}_k + \frac{\mathbf{F}(\mathbf{x}_{k+1}) \cdot (\mathbf{x}_{k+1} - \mathbf{x}_k)^T}{(\mathbf{x}_{k+1} - \mathbf{x}_k)^T \cdot (\mathbf{x}_{k+1} - \mathbf{x}_k)}$$

(both $\mathbf{x}$ and $\mathbf{F}(\cdot)$ are assumed to be *column* vectors). Use both Newton's method and Broyden's method to approximate the root $\mathbf{x}^* = [-1, -1]^T$ of the equation $\mathbf{F}(\mathbf{x}) = \mathbf{0}$ using the initial guesses $x_0 = [1.5, -1.5]^T$ and $\mathbf{D}_0 = \mathbf{I}$ (the identify matrix). Then,

(a) plot the approximation errors $\|\mathbf{x}_k - \mathbf{x}^*\|_2$ obtained with the two approaches as functions of k; this plot should use the logarithmic scaling for the vertical axis and appear as `figure(1)`,

(b) plot the all the approximations $\mathbf{x}_k$, $k = 0, 1, \dots$, produced by the two approaches in the (x_1, x_2) coordinates; make the region shown on this plot big enough so that *all* points $\mathbf{x}_k$ are visible also mark the exact solution $\mathbf{x}^*$; this plot should appear as `figure(2)`.

You may want to use the script `script_multi_Newton.m` posted on the course website as the starting point for your implementation.
(3 points)

2. Consider Runge's function

$$f(x) = \frac{1}{1 + a^2\, x^2}, \quad x \in [-1, 1],$$

which depends on a parameter $a > 0$, and its truncated Taylor-series expansion

$$g_N(x) = \sum_{k=0}^{N} a_k x^k, \quad x \in [-1, 1],$$

where $a_0, \dots, a_N$ are suitable coefficients and $N = 10$. Construct `chebfun` objects representing $f(x)$ and $g_N(x)$ and then determine by trial-and-error two values of the parameter a not too different from 1 such that

(a) the Taylor series $g_n(x)$, $n = 0, 1, \dots, N$, is uniformly convergent to $f(x)$ as n increases; plot both $f(x)$ and $g_n(x)$, $n = 0, 1, \dots, N$ as functions of x in `figure(3)` and save the value of a in the variable `Answer1`,

(b) the Taylor series $g_n(x)$, $n = 0, 1, \dots, N$, is *not* uniformly convergent to $f(x)$ as n increases; plot both $f(x)$ and $g_n(x)$, $n = 0, 1, \dots, N$ as functions of x in `figure(4)` and save the value of a in the variable `Answer2`.

In both cases compute the approximation error $\max_{-1 \le x \le 1} |f(x) - g_n(x)|$ and plot it as a function of the number of terms $n = 0, \dots, N$ in `figure(5)`. This plot should use the logarithmic scaling for the vertical axis, whereas the expression $\max_{-1 \le x \le 1} |\cdot|$ can be evaluated using `chebfun` function `norm(., inf)`.
(4 points)

3. Your are given the function $f(x) = x \cos(x^2)$ in the interval $\Omega = [-\pi, \pi]$ which is discretized using a uniform mesh with $N + 1$ grid points (take $x_0 = -\pi$ and $x_N = \pi$).

(a) consider the Vandermonde approach to interpolation and determine the value of N for which the condition number κ (obtained using MATLAB function `cond`) of the interpolation matrix exceeds the threshold $E = 10^6$; write out this value of N,

(b) calculate the Lagrange interpolating polynomials in the case when $N = 4$ and plot them using different colors and the step size $h = x_{i+1} - x_i = \frac{\pi}{50}$ in `figure(6)`,

(c) plot the function $f(x)$ together with its interpolants constructed using the Lagrange polynomials with $N = 4$ (in `figure(7)`) and $N = 16$ (in `figure(8)`); all functions should be plotted for $x \in \Omega$ using the same step size h as above; mark the interpolation points $\{x_i, f(x_i)\}_{i=0}^{N}$ with symbols.

(3 points)