HOMEWORK #5

Due: April 6 (Thursday) by midnight

Instructions:

- The assignment consists of *three* questions, worth 3, 4 and 3 points.
- Submit your assignment *electronically* to the Email address `math3q03@math.mcmaster.ca`; hardcopy submissions will not be accepted.
- It is obligatory to use the MATLAB template file available at `http://www.math.mcmaster.ca/bprotas/MATH3Q03/template.m` (see also the link in the "Computer Programs" section of the course website); submissions non compliant with this template will not be accepted.
- All plots should have suitable axis labels and legends.
- Make sure to enter your name and student I.D. number in the appropriate section of the template.
- Late submissions and submissions which do not comply with these guidelines will not be accepted.

---

1. You are given the following three functions, all defined for $x \in [-1, 1]$,

$$f(x) = \begin{cases} 0, & x \le 0, \\ x^5, & x > 0 \end{cases},$$

$$g(x) = \exp\left(\frac{-1}{1 - x^8}\right),$$

$$h(x) = \frac{1}{1 + \frac{1}{4}x^2}.$$

For each of these functions obtain the Chebyshev coefficients a_k, $k = 0, \dots, N$, where $N + 1$ is the number of coefficients computed by `chebfun` (which may be different for each of the functions). Plot $|a_k|$ versus k for all three functions using log-log and linear-log scaling of the axes, respectively, in `figure(1)` and `figure(2)`. Use symbols, rather than lines, to indicate the values of $|a_k|$. What rate of decay (algebraic, exponential or other) do we observe for each of the three functions? Print your answer (in the form of a short comment) on the screen and also record it (as a string) in variables `Answer1a`, `Answer1b` and `Answer1c`.
(3 points)

2. To quantify the rates decay of Chebyshev coefficients observed in the previous point, when possible, fit the relation $|a_k|$ versus k with

   - an algebraic law $\quad |a_k| \sim C\, k^{\alpha}$, or
   - an exponential law $\quad |a_k| \sim C\, 10^{\alpha k}$,

   for $k \geq k_0$ (k_0 is the index of the Chebyshev coefficient from which the law is assumed to hold). In either case the parameters C and α can be determined using a least-squares approximation, which can be conveniently done using MATLAB's function `polyfit`. Print the complete form of the fit (as a short comment) on the screen and also record it (as a string) in variables `Answer2a` and `Answer3b`. In addition, indicate the fits using dashed lines in `figure(1)` and `figure(2)`, as appropriate.
   (4 points)

3. You are given the *cardinal Whittaker* function $f(x) = \frac{\sin(x)}{x}$ on the interval $\Omega = [\pi/2, \pi]$.

   (a) Construct a family of *uniform* discretizations of the interval Ω using N grid points, where $N = 4, 8, 16, \ldots, 2097152$. On each of these grids approximate the derivative $f'(x)$ using the following finite-difference methods:

   i. first-order forward differences,
   ii. second-order central differences,
   iii. complex step derivative, i.e., $f'(x) = \frac{\Im(F(x+ih))}{h} + \mathcal{O}(h^3)$ where $F(z)$ is a complex analytic extension of $f(x)$, i.e., $F\ : \mathbb{C} \to \mathbb{C}$ and $F(x) = f(x)$ for $x \in \mathbb{R}$,
   iv. fourth-order central differences.

   The approximate derivative $f'_N(x_i)$ should be evaluated at *all* points x_i, $i = 1, \ldots .N$, in the given grid. For simplicity, when evaluating the finite-difference formulas at, or near, the endpoints x_1 and x_N you may use function values at points outside the interval Ω. For each of the methods compute the relative global approximation error of the derivative defined as

   $$E_N = \max_{1 \leq i \leq N} \left| \frac{f'_N(x_i) - f'_{exact}(x_i)}{f'_{exact}(x_i)} \right|$$

   and plot it on a log-log graph as a function of N. This plot should appear as `figure(3)`. You *need not* use `chebfun` for this part.

   (b) Construct a family of *nonuniform* discretizations of the interval Ω using N Chebyshev grid points ξ_i, $i = 1, \ldots, N$, where $N = 4, 8, 16, 32, 64$. On each of these grids approximate the derivative $f'(x)$ by first constructing a degree-($N - 1$) Chebyshev interpolating polynomial `pn` and then differentiating it analytically (which can be done in `chebfun` using the command `diff(pn)`). Evaluate thus obtained approximate derivatives at the Chebyshev points ξ_i, $i = 1, \ldots, N$, and plot the corresponding relative global approximation errors E_N as function of N in `figure(3)`.

   (3 points)