<u>Orthonormal polynomials</u> are obtained by imposing a particular normalization, i.e.,

$$(p_j, p_k)_w = \delta_{jk} = \begin{cases} 1, & k = j \\ 0, & k \neq j \end{cases}$$

<u>Examples</u>

* domain $= [-\pi, \pi]$ (periodic), $w(x) = 1$

  $p_k(x) = e^{ikx}$ (trigonometric functions in disguise)

* domain $= [-1, 1]$, $w(x) = 1$

  $p_k(x) = \sqrt{\frac{2k+1}{2}} \frac{1}{2^k k!} \frac{d^k}{dx^k} (x^2-1)^k$ – Legendre polynomials

* domain $= [-1, 1]$, $w(x) = \frac{2}{\pi\sqrt{1-x^2}}$

  $p_k(x) = \cos(k \arccos(x))$ – Chebyshev polynomials

  $T_0(x) = 1,\ T_1(x) = x,\ T_2(x) = 2x^2 - 1, \ldots$

There are many more families of orthogonal polynomials. Below we will focus mostly on Chebyshev polynomials.

<u>Ex</u> orth_poly_m

Chebyshev Minimal Amplitude Theorem

Of all polynomials of degree n with the leading coefficients equal to 1, the unique polynomial which has the <u>smallest</u> maximum on $[-1,1]$ is $T_n(x)/2^{n-1}$. In other words

$$\max_{x\in[-1,1]} |p_n(x)| \geq \max_{x\in[-1,1]} \left|\frac{T_n(x)}{2^{n-1}}\right| = \frac{1}{2^{n-1}}$$

Corollary

Consider the interpolation error

$$E(x) = \frac{f^{(n+1)}(\xi)}{(n+1)!} \underbrace{\prod_{i=0}^{n} (x-x_i)}, \quad \xi \in [-1,1]$$

for this factor to be the smallest possible, it must be given by a Chebyshev polynomial

$\Rightarrow x_i$ must be roots of $T_{n+1}(x)$

$$\prod_{i=0}^{n} (x-x_i) = \frac{1}{2^n} T_{n+1}(x)$$

Thus, the roots of the Chebyshev polynomial $T_{n+1}(x)$ are optimal from the point of view of interpolation errors $\Rightarrow$ <u>Chebyshev interpolation</u>

* ~~[illegible]~~ — Chebyshev points

$$x_j = \cos\left(\frac{j\pi}{n}\right), \quad 0 \leq j \leq n$$

Related to roots of unity

$$z_j = e^{ij\pi/n}, \quad -n+1 \leq j \leq n$$

$$z_j^{2n} = e^{i2\pi j} = (e^0)^j = 1$$

Interpolants, Projections & Aliasing

(Trefethen, Chapter 4)

Suppose $f : [-1, 1] \to \mathbb{R}$ is Lipschitz continuous. It then has a Chebyshev series with coefficients a_k, $k = 0, 1, \ldots$

$$f(x) = \sum_{k=0}^{\infty} a_k T_k(x), \quad a_k = (f, T_k)_w = \int_{-1}^{1} f(x) T_k(x) \, w(x) \, dx$$

$k \geq 1$

a_0 – additional factor of $\frac{1}{2}$

There are two distinct ways of representing $f(x)$ approximately as a polynomial in P_n.

* interpolation on the Chebyshev points

$$p_n(x) = \sum_{k=0}^{n} c_k T_k(x)$$

where $p_n(x_i) = f(x_i)$, $x_i = \cos\left(\frac{j\pi}{n}\right)$, $0 \leq j \leq n$

* truncation of the Chebyshev series

$$f_n(x) = \sum_{k=0}^{n} a_k T_k(x)$$

Question: what is the relation between:

- the coefficients a_k and c_k, $k = 0, \ldots, n$
- the corresponding functions $f_n(x)$ and $p_n(x)$?

Aliasing – consider a Chebyshev grid with $(n+1)$ points; when evaluated on this grid, any Chebyshev polynomial T_N (no matter how big N is) is indistinguishable from a single Chebyshev polynomial T_m for some m $0 \le m \le n$.

Ex – aliasing_01.m

Theorem (Aliasing of Chebyshev Polynomials)
For any $n \geq 1$ and $0 \le m \le n$, the following Chebyshev polynomials take the same values on the $(n+1)$-point Chebyshev grid

$$T_m,\ T_{2n-m},\ T_{2n+m},\ T_{4n-m},\ T_{4n+m},\ T_{6n-m}, \ldots$$

Equivalently, for any $k \geq 0$, T_k takes the same value on the grid as T_m with

$$m = |\,(k+n-1) \pmod{2n} - (n-1)\,|$$

a number in the range $0 \le m \le n$.

Proof (the first assertion only; for the proof of the second assertion, see Trefethen's book)

Chebyshev polynomials are related to the complex monomials on the unit circle as

$$T_m(x) = \frac{1}{2}\left(z^m + z^{-m}\right) = \cos(m\theta)$$

$$z = e^{i\theta} \qquad \theta = \cos^{-1}(x)$$