MATH 3Q03 DAY CLASS: FINAL EXAMINATION

9am, December 21, 2015, Duration: **2.5 hours**

Instructor: Dr. B. Protas

LAST (FAMILY) NAME: ______________________________

FIRST (GIVEN) NAME: ______________________________

STUDENT NUMBER: ______________________________

- This exam paper consists of 4 pages including this one. You are responsible for ensuring that your copy of the exam paper is complete. Bring any discrepancy to the attention of the invigilator.
- There are 9 questions, each worth the indicated number of points, for a total of 45 points.
- All answers must be entered in the booklets provided using permanent ink.
- Useful formulas are provided on the last page.
- Make sure to indicate your **name** and your **student number** above.
- Only the McMaster Standard Calculator Casio FX991MS is allowed.

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1. Consider the problem of finding roots of the equation $g(x) = 0$, where $g : \mathbb{R} \to \mathbb{R}$ is a continuously differentiable function. Assume one root belong to the interval $[a, b]$ for some $a < b$. Out of the different approaches we have discussed,

   (a) which is the method guaranteed to converge regardless of the choice of the initial approximation (as long as it brackets the root)?

   (b) which method has the fastest rate of convergence? characterize the rate of convergence of this method.

   [4 points]

2. Given a smooth function $f : \mathbb{R}^n \to \mathbb{R}$, we consider the minimization problem

$$\min_{\mathbf{x}\in\mathbb{R}^n} f(\mathbf{x}). \tag{1}$$

   Propose a computational approach which can be employed to solve this problem by reducing it to root-finding in $\mathbb{R}^n$. Are all points $\mathbf{x}$ found in this way guaranteed to be local minimizers of $f(\mathbf{x})$? If the function f has more than one local minimum, what is the best strategy to find these different minima?
   [4 points]

3. You are given two data points $\{x_1, y_1\}$ and $\{x_2, y_2\}$, where $x_1 < x_2$. Construct an interpolating polynomial $p_1(x)$ such that $p_1(x_i) = y_i$, $i = 1, 2$, using

   (a) the direct approach (based on the Vandermonde matrix),

   (b) Lagrange polynomials.

   [6 points]

4. Given a set of data points $\{x_i, y_i\}_{i=1}^{N}$, where N is a large number, show how to solve the least-squares approximation problem using $y(x) = ae^{bx}$ as the approximating function (with parameters $a, b \in \mathbb{R}$). Note that you may need to introduce a suitable transformation of the data and the approximating function before the least-squares problem is set up. Can this problem still be solved if $y(x) = a + \frac{b}{x}$ instead (assuming that $x_i \neq 0$, $i = 1, \dots, N$)? (provide an answer addressing the possibility, without solving the new problem).
   [5 points]

5. Define the "Runge phenomenon" and explain how it manifests itself in polynomial interpolation. Then,

   (a) identify the properties of the interpolated function $f(x)$ which determine whether or not the Runge phenomenon occurs in a given interpolation problem,

   (b) explain how a proper choice of the interpolation points can help mitigate this problem; justify your answer by citing a suitable theorem.

   [5 points]

6. You are given Chebyshev polynomial $T_1(x)$ and a Chebyshev grid which has only three points $x_0 = -1$, $x_1 = 0$ and $x_2 = 1$. Find the lowest-degree Chebyshev polynomial $T_m(x)$ which is aliased to $T_1(x)$ on the given grid (write down explicitly the coefficients of this polynomial). Provide a sketch of $T_1(x)$ and its alias $T_m(x)$ identifying the points where the the two polynomials attain the same values.
   [5 points]

7. Your are given the following functions, all defined for $x \in [-1, 1]$

   (a) $f(x) = x^{20} + x^{15} + x^{10} + x^5 + 1$,

   (b) $f(x) = \begin{cases} -1, & x < 0 \\ 1, & x \geq 0 \end{cases}$,

   (c) $f(x) = |x|$,

   (d) $f(x) = \begin{cases} 0, & x < 0 \\ x^2, & x \geq 0 \end{cases}$.

   For each of the functions consider the corresponding interpolant $p_n(x)$ based on Chebyshev polynomials and Chebyshev interpolation points. In each case describe how the interpolation error $\|f - p_n\|_\infty$ depends on the degree n of the interpolating polynomial as $n \to \infty$. Justify your answers and illustrate them with sketches of the relation $\|f - p_n\|_\infty$ versus n for the different functions. Is interpolation convergent in all cases?
   [8 points]

8. You are given a smooth function $y = f(x)$. Consider a finite-difference approximation of the function derivative $f'(x)$ and

   (a) sketch (using the log-log coordinates) the generic curve representing the dependence of the differentiation error on the step size h,

   (b) indicate the ranges of h dominated by errors of different type and show how we can determine the order of accuracy α of the finite-difference formula from this curve.

   [5 points]

9. You are given a smooth function $f : \mathbb{R} \to \mathbb{R}$ and a uniform discretization of the x-axis $x_i = ih$, where i is an integer and $h > 0$. Consider a forward-difference and a central-difference approximation of the derivative of $f(x)$ at $x = 0$, i.e.,

$$f'(0) \approx \frac{f_1 - f_0}{h}, \tag{2a}$$

$$f'(0) \approx \frac{f_1 - f_{-1}}{2h}, \tag{2b}$$

   where $f_i = f(x_i)$. Demonstrate that for sufficiently small h formula (2b) always provides a more accurate approximation of $f'(0)$ than formula (2a).
   [3 points]

END OF EXAM QUESTIONS

USEFUL FORMULAS

- Lagrange interpolating polynomial and functions based on $\{x_i, y_i\}_{i=1}^n$:

$$p_n(x) = \sum_{k=0}^{n} y_k \Phi_k(x), \text{ where } \Phi_k(x) = \frac{\Pi_{i=0, i\neq k}^n (x - x_i)}{\Pi_{i=0, i\neq k}^n (x_k - x_i)}, \quad k = 0, \ldots, n,$$

- Error of polynomial interpolation based on $\{x_k, y_k\}_{k=1}^n$:

$$E = f(x) - p_n(x) = \frac{f^{(n+1)}(\xi)}{(n+1)!} \prod_{k=0}^{n} (x - x_k)$$

- Chebyshev polynomials:

$$T_0(x) = 1, \quad T_1(x) = x, \quad T_{n+1}(x) = 2xT_n(x) - T_{n-1}(x), \quad n \geq 1, \ x \in [-1, 1]$$

- Chebyshev polynomials aliased to $T_m(x)$ ($0 \leq m \leq n$) on a Chebyshev grid with $n + 1$ points:

$$T_m, \ T_{2n-m}, \ T_{2n+m}, \ T_{4n-m}, \ T_{4n+m}, \ T_{6n-m}, \ldots$$

THE END