TEST #2

1:30–2:20pm, March 16 (Thursday), 50 minutes, 10 points max
(no textbooks, no notes)

Write your name and student number on the top of this sheet.
Write your answers on the reverse side and/or attach additional sheets as necessary.

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1. Your are given the following functions
   (a) $f(x) = \sin(x), \quad x \in \mathbb{R}$,
   (b) $f(x) = \frac{1}{x}, \quad x \in [0, 1]$,
   (c) $f(x) = \begin{cases} 0, & x \in [-1, 0) \\ 1, & x \in [0, 1] \end{cases}$,
   (d) $f(x) = \sqrt{x^2}, \quad x \in [-1, 1]$.

   According to the Weierstrass Approximation Theorem, which of these functions can be approximated on its domain of definition to an arbitrary precision with a polynomial? In each case justify your answer!
   [2 points]

2. You are given the following data points: $(x_0, y_0) = (-1, 0)$, $(x_1, y_1) = (0, -1)$ and $(x_2, y_2) = (1, 2)$. Use Lagrange functions to construct an interpolating polynomial $p_n(x)$, such that $p_n(x_i) = y_i$, $i = 0, 1, 2$. The Lagrange functions are given by

$$\Phi_k(x) = \frac{\Pi_{i=0, i\neq k}^{n}(x - x_i)}{\Pi_{i=0, i\neq k}^{n}(x_k - x_i)}, \quad k = 0, \dots, n,$$

   and the corresponding interpolating polynomial is $p_n(x) = \sum_{k=0}^{n} y_k \Phi_k(x)$. You need *not* use the barycentric interpolation formula.
   [4 points]

3. What can we say about the convergence of polynomial interpolation (i.e., the behavior of the errors as the degree n of the interpolating polynomial increases) when the following two functions are interpolated on a *uniform (equispaced) grid*

   (a)
$$f(x) = \frac{1}{1 + \frac{1}{25}x^2}, \qquad x \in [-1, 1],$$

   (b)
$$f(x) = \frac{1}{1 + 25\,x^2}, \qquad x \in [-1, 1]?$$

   Justify your answer.
   [2 points]

4. Your are given two data points: $(x_1, y_1) = (1, 1)$ and $(x_2, y_2) = (2, 2)$. Formulate and solve the least-squares approximation problem for this data when the approximating function is a degree-0 polynomial $y(x) = p_0(x) = C$, i.e., it is given by a constant $C \in \mathbb{R}$.
   [2 points]