HOMEWORK #5

Due: March 29 (Thursday) by midnight

Instructions:

- The assignment consists of *two* questions, worth 4 points each.
- Submit your assignment *electronically* (via Email) to the address `math4q03@math.mcmaster.ca`; hardcopy submissions will not be accepted.
- It is obligatory to use the MATLAB template file available at `http://www.math.mcmaster.ca/~bprotas/MATH4Q03/template.m` (see also the link in the "Computer Programs" section of the course website on the left); submissions non compliant with this template will not be accepted.
- Make sure to enter your name and student I.D. number in the appropriate section of the template.
- Late submissions and submissions which do not comply with these guidelines will not be accepted.

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1. Consider the following Boundary Value Problem

$$\frac{d^2y}{dx^2} + 2x\frac{dy}{dx} = 2(1-3x^2)e^{-x^2} \qquad x \in [0,2],$$
$$y(0) = 0,$$
$$y(2) + \frac{dy}{dx}(2) = -8e^{-4}$$

   (a) Use second–order central difference formulas to approximate this equation on the interior nodes and first–order backward finite differences to approximate the boundary conditions; plot the numerical solution obtained using the step size $h = \Delta x = 0.04$; you can use the code `tridiag1` posted on the course website to solve the resulting tridiagonal system.

   (b) Using the exact solution $y(x) = x^2 e^{-x^2}$, plot the *relative* error of the approximate solution at the right endpoint $x = 2$ obtained discretizing the interval $[0,2]$ with $N+1$ grid points, where N takes the values from `logspace(1,4,4)` (use a log–log plot).

   (4 points)

2. Consider the following Boundary–Value Problem with Dirichlet boundary conditions:

$$\begin{cases} u + \Delta u = [(2-\pi^2)y\sin(\pi y) + 2\pi\cos(\pi y)]e^{-x} & \text{in } \Omega = [0,1]\times[0,1] \\ u = g & \text{on } \Gamma, \end{cases}$$

   where

$$g(x=0,y) = y\sin(\pi y), \quad g(x=1,y) = e^{-1}y\sin(\pi y), \quad g(x,y=0) = g(x,y=1) = 0.$$

   (a) Solve this problem using the 5–point stencil with $h = \Delta x = \Delta y = 0.1$. Use the standard MATLAB command "\" to solve the resulting algebraic problem. Plot the obtained solution using the command `contourf(...)`.

(b) Solve the same problem using the grid sizes $h = \{0.1,\ 0.05,\ 0.02,\ 0.01\}$ and for every value of h calculate the error $E(h) = \max_{i,j} |\tilde{u}_{i,j} - u_{ex}(x_i, y_j)|$, where $\tilde{u}_{i,j}$ is the numerical solution and $u_{ex}(x,y) = y\sin(\pi y)e^{-x}$ is the exact solution (x_j and y_j are the discrete coordinates corresponding to a given value of h). Plot this error on the same figure as used in point 1(b) above. Why do the errors decrease at a different rate in the two cases?

HINT — You can start by modifying the code `FpointDi1.m` *[posted as a part of "Elliptic PDEs (II)"].*
(4 points)