HOMEWORK #6

Due: April 12 (Thursday) by midnight

Instructions:

- The assignment consists of *two* questions, worth 4 points each.
- Submit your assignment *electronically* (via Email) to the address `math4q03@math.mcmaster.ca`; hardcopy submissions will not be accepted.
- It is obligatory to use the MATLAB template file available at `http://www.math.mcmaster.ca/~bprotas/MATH4Q03/template.m` (see also the link in the "Computer Programs" section of the course website on the left); submissions non compliant with this template will not be accepted.
- Make sure to enter your name and student I.D. number in the appropriate section of the template.
- Late submissions and submissions which do not comply with these guidelines will not be accepted.

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1. Consider the following initial–boundary value problem for $u = u(x,t)$:

$$\begin{cases} \dfrac{\partial u}{\partial t} = \dfrac{\partial^2 u}{\partial x^2} + f(x,t), & \text{for } (x,t) \text{ in } \Omega = [-1,1] \times [0,1], \\ u(x,0) = -x^2 + 1, & \\ u(-1,t) = 0, & \\ u(1,t) = 0, & \end{cases}$$

where the right–hand side function is $f(x,t) = 2x^2 e^{-2t}$. Using the step size $\Delta x = 0.1$

(a) implement the Crank–Nicolson method and plot the solution obtained over the time–space domain using the time step $\Delta t = 0.05$; display the result as a space–time surface plot using the command `surf(...)`,

(b) using the analytical solution given in the form $u_{ex}(x,t) = (-x^2 + 1)e^{-2t}$ calculate the errors $E_{\Delta t} = \max_{x \in [-1,1]} |u_{\Delta t}(x, t_{max}) - u_{ex}(x, t_{max})|$ at the time instant $t_{max} = 1.0$, where $u_{\Delta t}$ is a numerical solution obtained with $\Delta t \in \{0.001 \;\; 0.005 \;\; 0.01 \;\; 0.05 \;\; 0.1\}$; display the errors $E_{\Delta t}$ as a function of Δt using a log–log plot,

(c) repeat the same computations using the explicit and implicit schemes and plot the errors on the same figure as in point (b); why does the error behave differently in one case?

HINT — you can start by modifying the code `heat_01.m` *posted on the course website; for simplicity, you can use the command* `A\ b` *instead of* `tridiag(...)` *to solve the resulting algebraic problems.*
(4 points)

2. Consider the following initial–boundary value problem for the advection equation:

$$\begin{cases} \dfrac{\partial u}{\partial t} + c\dfrac{\partial u}{\partial x} = g(x,t), & \text{for } (x,t) \text{ in } \Omega = [0,1] \times [0,1], \\ u(x,0) = \cos(\pi x), & \\ u(1,t) = \cos(\pi(1 - ct)) + \sin(\pi t), & \end{cases}$$

where $c = -2$ and the right–hand side function is given by $g(x,t) = \pi(x + ct)\cos(\pi x t)$. Assuming that $\Delta x = 0.05$ and $\Delta t = 0.01$, solve this problem using:

(a) the first–order *explicit* method, i.e., forward difference in both time and space, and

(b) the first–order *implicit* method, i.e., backward difference in time and forward difference in space.

In both cases treat the boundary conditions consistently with the approximations used in the methods. Plot the solutions as space–time surface plots (i.e., using the function `surf(...)`) on two separate figures.

(c) For both methods plot the error defined as $E(t) = \max_{x\in[0,1]} |u(x,t) - u_{ex}(x,t)|$ as a function of time [the analytical solution is given by $u_{ex}(x,t) = \cos(\pi(x-ct)) + \sin(\pi x t)$]; then repeat the same calculations using the time step $\Delta t = 0.001$ and plot the errors in the same figure; comment why the errors do not decrease when Δt is refined.

(4 points)