MATH 741 – APPLIED MATHEMATICS I

1 Elements of the ODE theory

1.1 Existence of solutions and some preliminaries

We will consider autonomous differential equations (DE) in the form

$$\dot{x} = f(x)$$

where

$$x \in \mathbb{R}^n, \quad t \geq 0, \quad \dot{x} = \frac{d}{dt}x(t)$$

$$f : D \to \mathbb{R}^n$$

D – open, connected, nonempty subset of $\mathbb{R}^n$

(= "domain of definition")

Remark

Scalar differential equations of higher order ~~[illegible]~~ (n-th order) can be transformed to systems of n equations of the first order

Ex Harmonic oscillator $\ddot{x} + \sin x = 0$

$$\begin{cases} x_1 = x \\ x_2 = \dot{x} \end{cases} \Rightarrow \frac{d}{dt}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} x_2 \\ -\sin x_1 \end{pmatrix}$$

Remark

In autonomous systems, the RHS does not depend on the independent variable (time). Nonautonomous systems can be transformed to an autonomous form, although the resulting autonomous system may have a more complicated form

Ex

$$\dot{x} = f(x,t) \Rightarrow \begin{cases} x_1 = x \\ x_2 = t \end{cases} \Rightarrow \frac{d}{dt}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} f(x_1, x_2) \\ 1 \end{pmatrix}$$

Remark

Systems of time-dependent differential equations are often referred to as dynamical systems.

Def

A classical solution of an ODE on an open interval $I \in \mathbb{R}$ is a function $x \in C^1(I)$ such that $\forall_{t \in I} \; \dot{x} = f(x)$

The Cauchy Problem (initial - value problem, IVP)

$$\begin{cases} \dot{x} = f(x) & x_0 \in \mathbb{R}^n \text{ — the initial condition} \\ x(0) = x_0, & 0 \in I \text{ (the solution interval must contain } t=0) \end{cases}$$

Remark

The Cauchy problem has an equivalent integral form:

$$x(t) = x_0 + \int_0^t f(x(\tau))\,d\tau$$

Def

The set $\{x(t) : t \in I\} \in D \subseteq \mathbb{R}^n$ is called a trajectory or orbit in phase space $\mathbb{R}^n$

Ex

If $f(x) = Ax + b$, where $A: \mathbb{R}^n \to \mathbb{R}^n$ and $b \in \mathbb{R}^n$, then the ODE is a linear inhomogeneous system. If $b = \bar{0}$, it is a linear homogeneous system.

<u>Def</u>

The Cauchy problem is locally <u>well-posed</u> (in the sense of Hadamard) if

(i) the solution $x(t)$ exists for
$$\mathbb{R} \supset I \neq \emptyset \quad (0 \in I)$$

(ii) the solution is unique

(iii) the solution depends <u>continuously</u> on x_0 and on parameters of $f(x)$

If $I \equiv \mathbb{R}$, the problem is globally well-posed.

If a problem is not well-posed, it is then <u>ill-posed</u>.

<u>Remarks</u>

ODEs often model evolution of state variables in a physical (chemical, ~~[illegible]~~ biological, etc.) process. ~~[illegible]~~ In such dynamical systems future states are uniquely determined by the initial states x_0. Therefore ODEs which do not have properties (i) and (ii) above are not acceptable models of a physical process.

ODEs which are not well-posed, because they do not meet condition (iii) can still be relevant as models of actual processes.

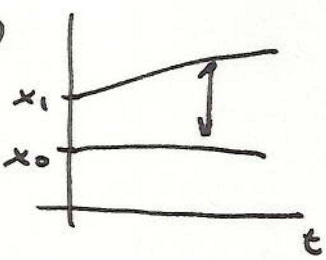
Some <u>explicit</u> examples

<u>Ex</u> $\begin{cases} \dot{x} = g(t) \\ x(0) = x_0 \end{cases}$ (an exception: non-autonomous ODE)

$$x(t) \equiv x(t; x_0) = x_0 + \int_0^t g(\tau)\,d\tau$$

① $x(t)$ exists $\forall_t$ if $g(t) \in C^0(\mathbb{R})$

② $y \stackrel{d}{\equiv} \underbrace{x_1 - x_2}_{\text{two possibly different sol'ns}}$: $\begin{cases} \dot{y} = 0 \\ y(0) = 0 \end{cases} \Rightarrow y(t) \equiv 0$ (uniqueness)

③ 

$$|x(t; x_1) - x(t; x_0)| = |x_1 - x_0|$$

(continuous dependence on the IC)

The ODE is <u>globally well-posed</u>.

<u>Ex</u> $\begin{cases} \dot{x} = x^2 \\ x(0) = x_0 > 0 \end{cases} \underset{\text{via separation of variables}}{\Longrightarrow} x(t) \equiv x(t; x_0) = \dfrac{x_0}{1 - t x_0}$

① $x(t)$ exists for $t < 1/x_0$

② $y \triangleq x_1 - x_2$: $\begin{cases} \dot{y} = (x_1 + x_2)\,y \\ y(0) = 0 \end{cases}$

Sol'n $\quad y(t) = \left[e^{\int_0^t (x_1(\tau) + x_2(\tau))\,d\tau} \right] y(0) \equiv 0$

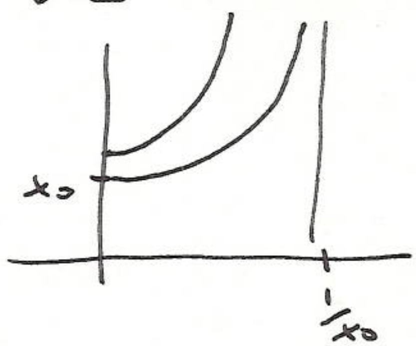
(uniqueness)

③ $|x(t;x_1) - x(t;x_0)| = \dfrac{x_1}{1 - t x_1} - \dfrac{x_0}{1 - t x_0}$

$$= \frac{(x_1 - x_0)}{1 - t(x_1 + x_0) + t^2 x_1 x_0} \qquad \text{(Magic)}$$

Smooth ~~dependence~~ dependence on x_0 for $t < \min(1/x_0, 1/x_1)$

The ODE is locally well-posed, but not globally well-posed, since $x(t;x_0) \to \infty$ as $t \to \left(\frac{1}{x_0}\right)^-$



An example of a "blow-up" solution.

<u>Ex</u>

$\begin{cases} \dot{x} = x^{1/2} \\ x(0) = x_0 \end{cases} \Rightarrow \begin{matrix} x(t) \equiv 0 \\ x(t) = \frac{1}{4} t^2 \end{matrix} \Rightarrow$ two distinct solution to the same Cauchy problem

The ODE is <u>not</u> locally well-posed. Why? ⑥

Ex Rössler system

$$\begin{cases} \dot{x} = -(y+z) \\ \dot{y} = x + ay \\ \dot{z} = b + xz - cz \end{cases} \qquad a,b,c \in \mathbb{R} \text{ - parameters}$$

Sensitive dependence on initial conditions
Numerical solution from Maple worksheet Chap7.mw
Observed behavior

$$|x(t;x_1) - x(t;x_0)| \leq C\,|x_1 - x_0|$$

$$C \sim e^{\alpha t}$$

Few problems can be solved analytically. We need a general theory that will allow us to determine whether a given ODE is well-posed without having to solve it analytically.

Before we will review some basic results from analysis.

Def (norm)
Let X be a vector space over the real numbers $\mathbb{R}$. A function $\|\cdot\|: X \to \mathbb{R}^+ = [0,\infty)$ is said to be a norm if:

(i) $\forall_{x \in X}\ \|x\| \geq 0, \quad \|x\| = 0 \iff x = 0$ (null element)

(ii) $\forall_{\alpha \in \mathbb{R}} \forall_{x \in X} \quad \|\alpha x\| = |\alpha| \|x\|$

(iii) $\forall_{x,y \in X} \quad \|x+y\| \leq \|x\| + \|y\|$

Remark

Norm has obvious generalization for vector spaces over the field of complex numbers $\mathbb{C}$

Ex for $x \in \mathbb{R}^n$

$$\|x\|_p = \left(\sum_{i=1}^{n} |x_i|^p \right)^{1/p}, \quad 1 \leq p < \infty$$

$$\|x\|_\infty = \max_{1 \leq i \leq n} |x_i|$$

Def

Normed vector space is a pair $(X, \|\cdot\|)$, where X is a vector space and $\|\cdot\|$ a norm on X.

Remark

Norm induces a notion of "distance" (topology) on X. ∎

Let $\{f_n\}$ be a sequence of real-valued functions defined on $D \subset \mathbb{R}^n$