Def

The sequence $\{f_m\}$ is called a uniform Cauchy sequence if

$$\forall_{\varepsilon>0} \; \exists M(\varepsilon)\in\mathbb{Z}, \text{ ST for } m>k\geq M$$

one has $|f_m(x) - f_k(x)| < \varepsilon$ $\forall_{x\in D}$

Def

A Banach space is a vector space X over real or complex numbers endowed with the norm $\|\cdot\|$, ST every Cauchy sequence (w.r.t the norm $\|\cdot\|$) is convergent (has a limit in X).

Banach space = complete normed vector space

Thm

The linear space $C(I)$ of functions continuous on the (closed) interval I and endowed with the norm

$$\|f\| = \sup_I |u(t)|$$

is a Banach space.

Def

The sequence $\{f_m\}$ is said to converge uniformly on D to a function f if

$\forall_{\varepsilon>0} \; \exists M(\varepsilon) \in \mathbb{Z}$ ST when $m > M$

$$|f_m(x) - f(x)| < \varepsilon \text{ uniformly } \forall_{x \in D}$$

ADDITION ↘

Suppose $T: X_1 \to X_2$ is a linear operator.

Def

The operator norm of T is defined as

$$\|T\| = \max_{\|x\|_{X_1} \leq 1} \|T(x)\|_{X_2} = \max_{\substack{x \in X_1 \\ x \neq 0}} \frac{\|T(x)\|_{X_2}}{\|x\|_{X_1}}$$

Ex

Consider $A: \mathbb{R}^n \to \mathbb{R}^n$, then $\|A\|_p = \max\limits_{x \neq 0} \dfrac{\|Ax\|_p}{\|x\|_p}$

$$\|A\|_1 = \max_{1 \leq j \leq n} \sum_{i=1}^{n} |a_{ij}|$$

$$\|A\|_\infty = \max_{1 \leq i \leq n} \sum_{j=1}^{n} |a_{ij}|$$

$$\|A\|_2 = \sqrt{\lambda_{\max}(A^* A)} = \sigma_{\max}(A)$$

A^* - conjugate transpose of A

$\lambda_{\max}$ - the largest eigenvalue of $A^* A$

Some properties of operator norms ($T, S : X \to X$)

(i) $\|T(x)\| \leq \|T\| \cdot \|x\|$

(ii) $\|TS\| \leq \|T\| \, \|S\|$

(iii) $\|T^k\| \leq \|T\|^k$, $k = 0, 1, 2, \ldots$

ADDITION

Def

Let $\mathcal{F}$ be a family of real-valued functions defined on a set $D \subset \mathbb{R}^n$. Then

(i) $\mathcal{F}$ is called ~~...~~ uniformly bounded if there is a $M > 0$ ST $|f(x)| \leq M \; \forall_{x \in D} \; \forall_{f \in \mathcal{F}}$

(ii) $\mathcal{F}$ is called equicontinuous on D if ~~...~~ $\forall_{\varepsilon > 0} \; \exists \, \delta > 0$ (independent of x, y, f) ST

$$|x - y| < \delta \Rightarrow |f(x) - f(y)| < \varepsilon \quad \forall_{x, y \in D} \; \forall_{f \in \mathcal{F}}$$

Ascoli-Arzela Lemma (important property of equicontinuous families of functions)

Let D be a closed, bounded subset of $\mathbb{R}^n$ and let $\{f_m\}$ be a sequence of real - valued functions in $C(D)$. If $\{f_m\}$ is equicontinuous and uniformly bounded on D, then there is a subsequence $\{m_k\}$ and a function f in $C(D)$ such that $\{f_{m_k}\}$ converges to f uniformly on D.

Proof: See Miller & Michel, p. 42, or any standard text on real analysis.

Peano's thm of existence of solutions will be proved in the following two steps:

* proof of existence of ε-approximate sol'ns
* demonstrate that ε-approximate sol'ns satisfy the original Cauchy problem

Def

An ε-approximate solution of

$$(*) \quad \begin{cases} \dot{x} = f(x) \\ x(0) = x_0 \end{cases}$$

on $I \supset 0$ is a real-valued function ϕ piecewise C^1 on I ST $\phi(0) = x_0$ and

$\forall_{t \in I}$ $\phi(t) \in D$ which satisfies

$$\| \dot{\phi}(t) - f(\phi(t)) \| < \varepsilon$$

at all points $t \in I$ where $\dot{\phi}(t)$ exists.

Let $x \in \overline{B}_\delta(x_0)$ [(closure of) the ball of radius δ with the center at x_0]

$$\left.\begin{array}{l} f \text{ - continuous} \\ \overline{B}_\delta(x_0) \text{ - compact} \\ \quad \text{(closed \& bounded)} \end{array}\right\} \begin{array}{l} f \text{ is } \underline{\text{bounded}} \text{ on } \overline{B}_\delta(x_0) \text{, i.e.} \\ \exists M > 0, \text{ ST } \|f(x)\| \leq M \ \forall_{x \in B_\delta(x_0)} \end{array}$$

Let $\tau = \delta / M$

<u>Lemma</u>

For any $\varepsilon > 0$, $\exists\, \delta(\varepsilon) > 0$, $\tau(\varepsilon) > 0$ ST there exists an ε-approximate solution ϕ of (*) on the interval $[-\tau, \tau]$.

<u>Proof</u>

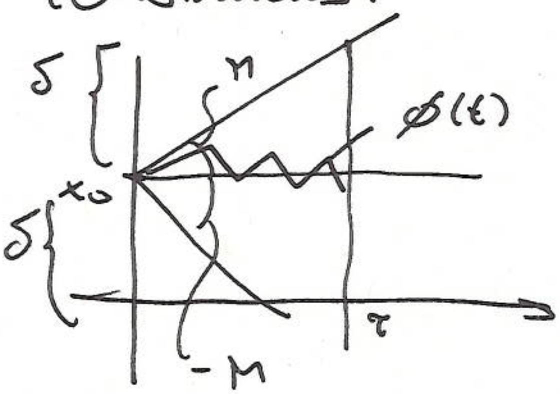
We'll consider the ~~proof~~ interval $[0, \tau]$. The proof for $[-\tau, 0]$ is similar. The proof relies on Euler's method of numerical analysis.

Divide $[0,\tau]$ into m equal subintervals $[0,t_1], [t_1,t_2], \ldots, [t_{m-1},\tau]$

Let

$$\phi(t) = \phi(t_j) + g(\phi(t_j))(t-t_j) \quad \text{for } [t_j, t_{j+1}]$$

Thus, $\phi(t)$ is piecewise linear and continuous.



It is clear that $\phi(t) \in \bar{B}_\delta(x_0)$ $\forall\, t \in [0,\tau]$

Since

$$\|\phi(t) - x_0\| \leq m M \frac{\tau}{m} = M\tau = \delta$$

Then we have

$$\|\dot{\phi}(t) - g(\phi(t))\| = \|g(\phi(t_j)) - g(\phi(t))\| < \varepsilon$$

which is true),

since

$$\|\phi(t) - \phi(t_j)\| \leq M(t-t_j) \leq M \frac{\delta}{M} = \delta$$

because, by the continuity of g, we have

$$\forall\, \varepsilon > 0 \;\; \exists\, \delta(\varepsilon) > 0 \text{ ST } \|x - y\| < \delta \Rightarrow \|g(x) - g(y)\| < \varepsilon$$

Thm (Peano)

If $f \in C(D)$ and $x_0 \in D$, then problem (*) has a solution defined on $[-\tau, \tau]$

Proof

Let $\{\varepsilon_m\}_{m \in \mathbb{N}}$ be a sequence decreasing monotonously to zero., and $\{\phi_m(t)\}_{m \in \mathbb{N}}$ be the corresponding ε_m-approximate solutions ST

$$E_m(t) \triangleq \dot{\phi}_m(t) - f(\phi_m(t)) \quad \forall$$

$$\|E_m(t)\| \leq \varepsilon_m \qquad t \in [\tau, \tau]$$

therefore

$$\phi_m(t) = x_0 + \int_0^t f(\phi_m(s))ds + \int_0^t E_m(s)ds$$

By construction $\|\phi_m(t) - \phi_m(s)\| \leq M_{\delta_m}|t-s| \ \forall_{t,s \in [-\tau,\tau]}$

and $M_{\delta_m} = \sup_{x \in B_{\delta_m}(x_0)} \|f(x)\|$

and then

$$\|\phi_m(t)\| \leq \|\phi_m(0)\| + \|\phi_m(t) - \phi_m(0)\|$$

$$\leq \|x_0\| + M_{\delta_m}\tau < \delta_m$$

Therefore $\{\phi_m(t)\}_{m\in\mathbb{N}}$ is an equicontinuous and bounded sequence of functions.
By Ascoli - Arzela Lemma, there is a subsequence $\{\phi_{m_k}(t)\}_{k\in\mathbb{N}}$ and a function $\phi(t)$ such that $\phi_{m_k}(t) \xrightarrow[k\to\infty]{} \phi(t)$ uniformly on $[-\tau,\tau]$.
Since $g(\phi)$ is uniformly continuous, then

$$\alpha_{m_k} \triangleq \sup_{t\in[-\tau,\tau]} \|g(\phi_{m_k}(t)) - g(\phi(t))\| \xrightarrow[k\to\infty]{} 0$$

Therefore

$$\|\phi_{m_k}(t) - \underbrace{x_0 - \int_0^t g(\phi(s))ds}_{\phi(t)}\| \leq$$

$$\int_0^t \|g(\phi(s)) - g(\phi_{m_k}(s))\|ds + \int_0^t \|E_{m_k}(s)\|ds$$

$$\leq (\alpha_{m_k} + \varepsilon_{m_k})\,\tau \xrightarrow[k\to 0]{} 0.$$

Thus, $\phi(t)$ solves $\phi(t) = x_0 + \int_0^t g(\phi(s))ds$, and $\phi(t) \equiv x(t)$

■

<u>Remark</u>

Uniqueness is not ensured.