1.3 Continuous Dependence on Initial Conditions

Continuity of solutions w.r.t initial data is impossible if there is no uniqueness. In fact, is not only necessary, but also sufficient for uniqueness.

Theorem Let $\delta > 0$ and $y \in B_\delta(x_0) \subset D$. Let $f(D)$ be Lipschitz continuous. Let $x = x(t;y)$ be a unique solution of

$$\begin{cases} \dot{x} = f(x) \\ x(0) = y \end{cases} \quad \text{for } t \in I(y) = [-\tau(y), \tau(y)]$$

Then,

$$\forall_{x_0 \in D} \lim_{y \to x_0} \|x(t;y) - x(t;x_0)\| = 0$$

uniformly on $t \in I_0 = [-\tau_0, \tau_0]$ where $\tau_0 = \inf_{y \in B_\delta(x_0)} (\tau(y))$

Proof

By previous theorems, two solutions $x(t;y)$ and $x(t;x_0)$ exist and are unique for $t \in [0, \tau_0]$. Then

$$\|x(t;y) - x(t;x_0)\| \leq \|y - x_0\| + \int_0^t \|f(x(s;y)) - f(x(s;x_0))\| ds$$

$$\leq \|y - x_0\| + K_0 \int_0^t \|x(s;y) - x(s;x_0)\| \, ds, \quad \text{where } \forall t \in [0, \tau_0]$$

where

$$\|g(y_1) - g(y_2)\| \leq K_0 \|y_1 - y_2\| \quad \forall y_1, y_2 \in B_\delta(x_0)$$

By Gronwall's Lemma:

$$\|x(t;y) - x(t;x_0)\| \leq \|y - x_0\| e^{K_0 \tau_0}$$

so that

$$\lim_{y \to x_0} \|x(t;y) - x(t;x_0)\| = 0$$

uniformly for $t \in I_0$ ∎

<u>Remark</u>

Continuous parameter dependence can be studied by consider a Cauchy problem for an extended system.

<u>Ex</u>

$$\begin{cases} \dot{x} = g(x, \lambda_0) \\ x(0) = x_0 \end{cases} \xrightarrow[\text{extended system}]{} \begin{cases} \dfrac{d}{dt}\begin{pmatrix} x \\ \lambda \end{pmatrix} = \begin{pmatrix} g(x,\lambda) \\ 0 \end{pmatrix} \\ x(0) = x_0 \\ \lambda(0) = \lambda_0 \end{cases}$$

Remark

Generally speaking, the solution $x(t;y)$ is as smooth in y as the vector field $f(x)$ at $x=y$. For instance, if $f \in C^k(D)$, then $x(t;y)$ is C^k in y for any $y \in D$. If f is real-analytic in D, then $x(t;y)$ is real-analytic in y.

Theorem (smooth dependence on initial data)

Let $f \in C^1(D)$ and $x(t;y)$ be a sol'n of the Cauchy problem for $y \in D$. Then

$$\forall_{x_0 \in D} \; \exists \, \tau > 0, \varepsilon > 0 \quad \text{ST} \quad x(t;y) \text{ is } C^1 \text{ in } y \; \forall_{y \in B_\varepsilon(x_0)} \text{ on } I = [-\tau, \tau]$$

Proof

We have the map:

$$x(t;\cdot) : D \longrightarrow C^1(I)$$

Need to show that this map is C^1

Def (Fréchet derivative)

Let V, W - Banach spaces and $U \subset V$.
A function $f : U \to W$ is Fréchet differentiable

if there exists a bounded linear operator
$\phi_x : V \to W$ such that

$$\lim_{h \to 0} \frac{\| f(x+h) - f(x) - \phi_x h \|_W}{\| h \|_V} = 0$$

To show differentiability of $x(t;y)$ w.r.t we need to demonstrate the existence of a suitable Fréchet derivative

Consider the Jacobian matrix $D_x f(x)$ which continuous in $x \in D$ $\left\{ [D_x f(x)]_{ij} = \frac{\partial f_i(x)}{\partial x_j} \right\}_{i,j=1,\dots,n}$

Let $\phi(t) \in \mathbb{R}^{n \times n}$ be a matrix solution of the linearized system

$$\begin{cases} \dfrac{d}{dt}\phi(t) = D_x f(x(t;x_0))\,\phi(t) \\ \phi(0) = I \quad \text{(identity matrix)} \end{cases}$$

then we have

$$\|\phi(t)\| \leq \|I\| + \int_0^t \| D_x f(x(s,x_0))\phi(s) \| ds$$

$$\leq 1 + \int_0^t \| D_x f(x(s,x_0)) \| \cdot \|\phi(s)\| ds$$

$$\leq 1 + K_0 \int_0^t \|\phi(s)\| ds$$

where $K_0 = \sup_{t \in I} \| D_x f(x(t;x_0)) \|$, $x(t,x_0) \in D$ for $t \in I$

Therefore, by Gronwall Lemma

$$\|\phi(t)\| \leq 1 \cdot e^{K_0 t} \quad \forall_{t \in I}$$

Thus, $\phi(t)$ is bounded for all times (it is also unique).

Next consider ($h = y - x$)

$$x(t;y) - x(t;x_0) - \phi(t)(y - x_0)$$

$$= \int_0^t \left[f(x(s;y)) - f(x(s;x_0)) - D_x f(x(s;x_0)) \phi(s)(y - x_0) \right] ds$$

$$= \int_0^t \left\{ D_x f(x(s;x_0)) \left[x(s;y) - x(s;x_0) - \phi(s)(y - x_0) \right] + R(s;x_0,y) \right\} ds$$

where we used Taylor's theorem:

$$f(x(t,y)) = f(x(t;x_0)) + D_x f(x(t,x_0)) \left[x(t;y) - x(t;x_0) \right] + \underbrace{R(t;x_0,y)}_{\text{Lagrange form of the remainder term}}$$

Note that

$$\lim_{\|y - x_0\| \to 0} \frac{\|R(t;x_0,y)\|}{\|y - x_0\|} = 0 \quad \text{uniformly on } I$$

Therefore, by Gronwall Lemma,

$$\|x(t;y) - x(t;x_0) - \phi(t)(y-x_0)\| \leq M_0 e^{K_0 \tilde{\tau}}$$

for all $t \in [0,\tilde{\tau}]$

where $M_0 = \sup_{t \in I} \|R(t;x_0,y)\| = M_0(x_0,y)$

(K_0 defined before)

~~[illegible]~~ Invoking now the definition of the Fréchet - differentiability

$$\cancel{\lim_{y \to x_0} \frac{\|x(t;y) - x(t;x_0) - \phi(t)(y-x_0)\|}{\|y-x_0\|}}$$

$$\lim_{y \to x_0} \frac{\sup_{t \in I} \|x(t;y) - x(t;x_0) - \phi(t)(y-x_0)\|}{\|y-x_0\|}$$

$$\leq e^{K_0 \tilde{\tau}} \cdot \lim_{y \to x_0} \frac{M_0(x_0,y)}{\|y-x_0\|}$$

$$= e^{K_0 \tilde{\tau}} \cdot \lim_{y \to x_0} \frac{\sup_{t \in I} \|R(t;x_0,y)\|}{\|y-x_0\|} = 0$$

(by the property of the Lagrange remainder term)

So that $x(t;y)$ is continuously differentiable in y at $y = x_0 \in D$ and

$$\phi(t) \equiv D_y x(t;y_0)\big|_{y=x_0}$$

Ex

$$\begin{cases} \dot{x} = -\frac{1}{2x} \\ x(0) = y > 0 \end{cases} \Rightarrow x(t;y) = \sqrt{y^2 - t} \quad \text{exists for any } y>0 \text{ and } t \le y^2$$

The solution is differentiable w.r.t y for any $y>0$, but not for $y=0$.

Ex

$$\begin{cases} \dot{x} = \lambda x \\ x(0) = y \end{cases} \Rightarrow x(t;y) = y e^{\lambda t} \quad - \text{ real-analytic in } y \in \mathbb{R}.$$