Ex 1

$$\begin{cases} \dot{x} = -\frac{1}{2x} \\ x(0) = y > 0 \end{cases} \Rightarrow x(t;y) = \sqrt{y^2 - t} \text{ exists for any } y>0 \text{ and } t \le y^2$$

The solution is differentiable w.r.t y for any $y>0$, but not for $y=0$.

Ex

$$\begin{cases} \dot{x} = \lambda x \\ x(0) = y \end{cases} \Rightarrow x(t;y) = y e^{\lambda t} \text{ - real-analytic in } y \in \mathbb{R}.$$

Ex 1 - cont

$$\phi(t) = D_y x(t;y)\big|_{y=x_0}$$

$$\begin{cases} \dot{\phi}(t) = \left(\frac{1}{2x^2(t;x_0)}\right)\phi(t) \\ \phi(0) = 1 \end{cases}$$

Ex 2 cont

$$\begin{cases} \dot{\phi}(t) = \lambda \phi(t) \\ \phi(0) = 1 \end{cases}$$ (same structure as the original ODE)

1.4 PHASE FLOWS & PHASE SPACE DEFINED BY DIFFERENTIAL EQUATIONS

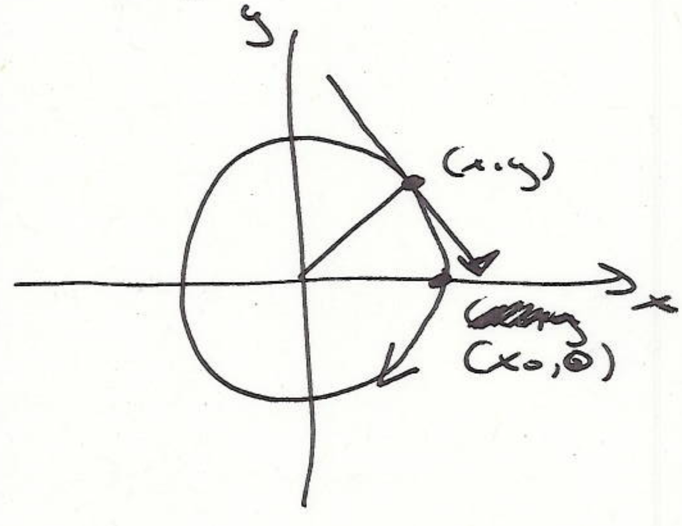
The solution $x = x(t; x_0)$ of the Cauchy problem $\begin{cases} \dot{x} = g(x) \\ x(0) = x_0 \end{cases}$ represents a trajectory (orbit) in phase space $\mathbb{R}^n$ passing through the point $x_0 \in \mathbb{R}^n$, such that the <u>vector field</u> is tangent to the curve at each point ($\Rightarrow$ geometric theory of dynamical systems)

<u>Ex</u>

$$\ddot{x} + x = 0 \Rightarrow \frac{d}{dt}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} y \\ -x \end{pmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$$

$$\begin{cases} x(0) = x_0 \\ y(0) = 0 \end{cases} \Rightarrow \begin{array}{l} x(t) = x_0 \cos t \\ y(t) = -x_0 \sin t \end{array}$$

A family of $\overbrace{\text{periodic}}^{\text{closed}}$ orbits (with period 2π) parametrized by the initial condition $(x_0, 0)$



<u>Ex</u> Maple file Chapter 2 (two examples in $\mathbb{R}^2$)

Maple file for Chapter 7 (two examples in $\mathbb{R}^3$)

Def

The mapping $I \times D \longrightarrow x(t; x_0) \triangleq \phi_t(x_0)$ on $t \in I \subset \mathbb{R}$ for any $x_0 \in D \subseteq \mathbb{R}^n$ is called the phase flow for the ODE

$$\begin{cases} \dot{x} = f(x) & \forall_t \in I \\ x(0) = x_0 & x_0 \in D \end{cases}$$

Ex

$$\begin{cases} \dot{x} = \lambda x \\ x(0) = x_0 \end{cases} \Rightarrow \phi_t(x_0) = e^{\lambda t} x_0 : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$$

(Linear in x_0 and entire in t)

Properties of Phase Flows

1) $\phi_0(x_0) = x_0$

Pf

Evident if a unique sol'n exists

2) ~~[illegible]~~

Thm

Let f be Lipschitz on D and assume that $\phi_t(x_0) \in D$ on $t \in [0, t_0 + t_1]$ for a given $x_0 \in D$ and some t_0, t_1 ST $t_0 + t_1 \in I$. Then

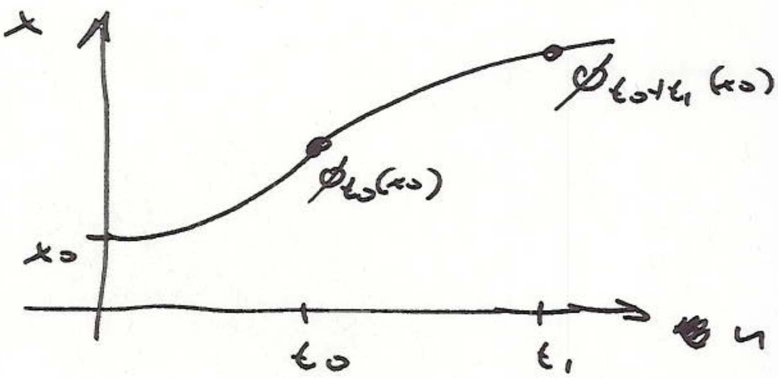
$$\phi_{t_1}(\phi_{t_0}(x_0)) = \phi_{t_0 + t_1}(x_0)$$

Proof

$$\begin{cases} \dfrac{dx}{du} = f(x) & \forall \; u \in [0, t_0+t_1] \\ x(0) = x_0 \end{cases} \qquad x(u) = \begin{cases} \phi_u(x_0), & 0 \le u \le t_0 \\ \phi_{u-t_0}(\phi_{t_0}(x_0)), & t_0 \le u \le t_0+t_1 \end{cases}$$

By the existence/uniqueness thm

$$x(t_0+t_1) = \phi_{t_0+t_1}(x_0) = \phi_{t_1}(\phi_{t_0}(x_0))$$



Remark

This property defines a group of all trajectories parameterized by $x_0 \in D$

Remark

If $f \in C^1(D)$, then the map $\phi_t(x_0)$ is differentiable in $x_0 \in D$

Property 3 [Corollary from Property 2]

Let $t_1 = -t_0$

Then $\phi_{-t_0}(\phi_{t_0}(x_0)) = \phi_0(x_0) = x_0$

Ex

$$\begin{cases} \dot{x} = x^2 \\ x(0) = x_0 \end{cases} \Rightarrow \phi_t(x_0) = \frac{x_0}{1 - t x_0}, \quad t < \frac{1}{x_0}$$

(P1) $\phi_0(x_0) = x_0$

(P2) $$\phi_{t_1}(\phi_{t_0}(x_0)) = \frac{\frac{x_0}{1 - t_0 x_0}}{1 - t_1 \frac{x_0}{1 - t_0 x_0}} = \frac{x_0}{1 - (t_0 + t_1) x_0}$$

$$= \phi_{t_0 + t_1}(x_0) \quad \forall \, t_0, t_1, \; t_0 + t_1 < \frac{1}{x_0}$$

(P3) $\phi_{-t}(\phi_t(x_0)) = x_0$

In many practical cases the group properties of $\phi_t(x_0)$ become much simpler when the maximal interval of existence is mapped to $\mathbb{R}$, so that it becomes independent of $x_0 \in D$

Thm

Let $x(t)$ be a (classical) solution of $\begin{cases} \dot{x} = g(x) \\ x(0) = x_0 \end{cases}$ on $[0, \tau_0]$ and $g(x) : D \to \mathbb{R}^1$ be a positive continuous function. Let

$u^{(t)} = \int_0^t \frac{ds}{g(x(s; x_0))}$. Then $y(u^{(t)}; x_0) \equiv x(t; x_0)$

is a solution of the Cauchy Problem

$$\begin{cases} \dfrac{dy}{du} = g(y)\, f(y) & \text{for } t \in \mathbb{R}^+ \\ y(0) = x_0 \end{cases}$$

Proof

$$\frac{du}{dt}(t) = \frac{1}{g(x(t;x_0))}$$

$$\frac{dy}{du} = \frac{dx}{dt}\frac{dt}{du} = f(x)\,g(x) = f(y)\,g(y)$$

$$y(0) = x(0;x_0) = x_0$$

Since $g(x) > 0$, $\dfrac{du}{dt} = \dfrac{1}{g(x)} > 0$ and $u(t;x_0)$ is invertible w.r.t $t \in [0, \tau_0]$

Remark

~~The vector fields~~ Due to the positivity of $g(x)$, the vector fields $f(x)$ and $f(x)\,g(x)$ have the same directions at each $x \in D$. Therefore, the ~~~~ topology of the phase portraits of the two ODEs is the same (i.e., they have topologically equivalent families of trajectories). The only difference is the "time" required to reach the corresponding points on the two phase portraits.

Ex

$$\begin{cases} \dot{x} = x^2 \\ x(0) = x_0 > 0 \end{cases}$$

Let $g(x) = \frac{1}{x} > 0$ if $x > 0$

Then
$$u(t) = \int_0^t \frac{ds}{\frac{1}{x(s;x_0)}} = \int_0^t x(s;x_0)\,ds$$

Then ~~xxx~~ $y(u) \equiv x(t)$

and

$$\begin{cases} \frac{dy}{du} = y \\ y(0) = x_0 \end{cases} \Rightarrow y(u;x_0) = x_0 e^{u}$$

$$\frac{du}{dt} = x(t;x_0) = x_0 e^{u} \Rightarrow \frac{dt}{du} = e^{-u} x_0^{-1}$$

integrating using the condition $u(t=0) = 0$

$$t = \frac{1}{x_0}\left(1 - e^{-u}\right) \Rightarrow e^{u} = \frac{1}{1 - t x_0} \Rightarrow x(t;x_0) = \frac{x_0}{1 - t x_0}$$

$$u \to \infty \Rightarrow t \to \left(\frac{1}{x_0}\right)^{-}$$
$$u \to -\infty \Rightarrow t \to -\infty$$

Thus, after rescaling, the blow-up is mapped to the infinite time.

II LINEAR SYSTEMS & STABILITY

2.1 PROPERTIES OF LINEAR SYSTEMS

Linear systems are ~~[illegible]~~ important:

* in their own right
* as linearizations of nonlinear dynamical systems around particular solutions

A <u>general</u> dynamical system is

$$\begin{cases} \dot{x} = A(t)\,x + b(t) \\ x(0) = x_0 \end{cases} \quad , \; x(t),\, b(t) \in \mathbb{R}^n$$

$A(t) : \mathbb{R}^n \to \mathbb{R}^n$

$t \in \mathbb{R}$

Homogenous linear systems

$$\begin{cases} \dot{x} = A(t)\,x \\ x(0) = x_0 \end{cases}$$

Important particular cases:

* $A(t) = A_0$ (linearization at constant solutions)

* $A(t+T) = A(t)$ (linearization at periodic [with period T] solutions)

Remark

Unique solutions $x(t) \in \mathbb{R}^n$ exist for $t \in I$ and $x_0 \in \mathbb{R}^n$ if $A(t)$ and $b(t)$ are continuous for $t \in I \subset \mathbb{R}$.

[Proof: via application of previous theorem generalized to the non-autonomous case]

Lemma (Liouville's formula)

Let $\phi(t)$ be an $n \times n$ matrix of any solutions of $\dot{x} = A(t)x$, i.e.

$$\begin{aligned} \dot{x}_1 &= A(t)x_1 \\ \dot{x}_2 &= A(t)x_2 \\ &\vdots \\ \dot{x}_n &= A(t)x_n \end{aligned} \Rightarrow \phi(t) = \left[x_1(t), x_2(t), \dots, x_n(t)\right]$$

$$\Rightarrow \dot{\phi} = A(t)\phi$$

Then

$$\det \phi(t) = \left[e^{\int_0^t \operatorname{tr} A(s)\,ds}\right] \det \phi(0)$$

Proof

From the equation we have

$$\forall_{t \in I} \lim_{h \to 0} \left[\frac{\phi(t+h) - \phi(t)}{h} - A(t)\phi(t)\right] = 0$$

$$\Rightarrow \phi(t+h) = \phi(t) + hA(t)\phi(t) + R_h(t), \text{ ST}$$

$$\lim_{h \to 0} \frac{R_h(t)}{h} = 0$$

uniformly for $t \in I$

$$= [I + hA(t)]\,\phi(t) + R_h(t)$$

|| $[I+hA(t)]$ - invertible for sufficiently small t

$$= [I + hA(t)]\left\{\phi(t) + [I+hA(t)]^{-1}R_h(t)\right\}$$

Then

$$\det\phi(t+h) = \det(I+hA(t))\,\det\Big(\phi(t)+\overbrace{[I+hA(t)]^{-1}R_h(t)}^{r_h(t)}\Big)$$

Use the following property of determinants:

$$\det(I+xA) = \sum_{k=0}^{\infty}\frac{1}{k!}\left[-\sum_{j=1}^{\infty}\frac{(-x)^j}{j}\,\mathrm{tr}(A^j)\right]^k$$
$$= \alpha_0+\alpha_1 x+\alpha_2 x^2+\dots$$

Thus

$$\det\phi(t+h) = \left[1+h\,\mathrm{tr}A(t)+h^2 q_h(t)\right]\det\big(\phi(t)+r_h(t)\big)$$

Rearranging and taking the limit

$$\lim_{h\to 0}\left[\frac{\det\phi(t+h)-\det(\phi(t)+r_h(t))}{h} - \mathrm{tr}A(t)\det(\phi(t)+r_h(t))\right.$$

$$\left. + h(\dots)\right]=0 \Rightarrow \frac{d}{dt}\big(\det\phi(t)\big) = \mathrm{tr}A(t)\cdot\det\phi(t)$$

$\forall\, t\in I$

Note that
$$\lim_{h\to 0}\frac{r_h(t)}{h}=0$$

∎

<u>Corollary</u>

Solutions $x_1(t), x_2(t), \dots, x_n(t)$ which are linearly independent at $t=0$, will remain linearly independent for all $t>0$.