Ex $\dot{x} = -x^3$ (from an earlier example ~~re~~ regarding linearized stability)

$V(x) = \frac{1}{4} x^4$

$$\left.\begin{aligned} &① \ V(0) = 0, \\ &② \ V(x) > 0 \quad \forall_{x \in \mathbb{R} \setminus \{0\}} \\ &③ \ \frac{dV}{dt} = x^3 \cdot \dot{x} = -x^6 \leq 0 \quad \forall_{x \in \mathbb{R}} \end{aligned}\right\} \Rightarrow x = 0 \text{ asymptotically stable}$$

Proof

Fix $\varepsilon > 0$ and look for $\delta(\varepsilon) > 0$ ST

$$\forall_{x_0 \in B_\delta(x_*)} \quad x(t) = \phi_t(x_0) \in B_\varepsilon(x_*) \quad \forall_{t \geq 0}$$

Let $m_\varepsilon = \min_{x \in \partial B_\varepsilon(x_*)} V(x) > 0$ (minimum over the boundary of $B_\varepsilon(x_*)$)

$$\left.\begin{aligned} &V \in C^1(B_\varepsilon(x_*)) \\ &V(x_*) = 0 \end{aligned}\right\} \Rightarrow \exists \delta > 0 \text{ ST } V(x) < m_\varepsilon \quad \forall_{x \in B_\delta(x_*)}$$

$\dot{V} \leq 0 \Rightarrow V(x(t))$ non-increasing function along solution $x(t)$

$$\Rightarrow \forall_{x_0 \in B_\delta(x_*)} \ \forall_{t \geq 0} \quad V(\phi_t(x_0)) \leq V(x_0) < m_\varepsilon$$

By contradiction, suppose $\exists t_1 \geq 0$ ST $\|\phi_t(x_0)\| = \varepsilon$ (on the boundary). Then $V(\phi_{t_*}(x_0)) \geq m_\varepsilon$. However, this is a contradiction with $V(\phi_t(x_0)) < m_\varepsilon$!
Thus, $\forall_{x_0 \in B_\delta(x_0)}, \forall_{t \geq 0}$ we have $\phi_t(x_0) \in B_\varepsilon(x_*)$ ∎

## Remark

If $\frac{dV}{dt} = 0 \quad \forall_{x(t)}$ ST $\frac{dx}{dt} = \underline{f}(x)$, then the trajectory $x(t)$ lies on a level set $V(x) = C$. The function $V(x)$ is called the ~~energy~~ energy of the system $\dot{x} = \underline{f}(x)$.

**Ex** $\begin{cases} \dot{x}_1 = -(x_1^3 + x_2^3) \\ \dot{x}_2 = -(x_1^3 + x_2^3) \end{cases}$

Let: $V(x_1, x_2) = \frac{1}{4}(x_1^4 + x_2^4) > 0$

$\forall_{(x_1, x_2) \in \mathbb{R}^2 - \{0\}}$

$V(0,0) = 0$

$$\frac{dV}{dt} = x_1^3 \dot{x}_1 + x_2^3 \dot{x}_2$$
$$= -(x_1^3 + x_2^3)^2 < 0$$
$$\Rightarrow (0,0) - \underline{\text{stable}}$$

## Remark

There is no specific rule telling how to construct Lyapunov function $V(x)$, or whether such a function at all exists. This is a major shortcoming of the Lyapunov method. In many problems one can try to work with quadratic forms

$$V(x_1, \ldots, x_n) = \sum_{i=1}^{n} \sum_{j=1}^{n} b_{ij} x_i x_j, \text{ where}$$

$[b_{ij}]$ is a $n \times n$ real symmetric matrix.

Theorem (Lyapunov Instability Theorem)

Suppose there exists a $C^1$ function $V(x)$ such that:

① $V(0) = 0$

② $\forall_{\delta>0} \; \exists \, x_0 \neq 0, \; \|x_0\| < \delta$ ST $V(x_0) > 0$

③ $\frac{d}{dt} V(x) > 0$ in $U$, where $U = \{x \in B_r, V(x) > 0\}$ for some $r > 0$ and $B_r \subset D$

Then, the critical point $x = 0$ is unstable.

Proof

The set $U$ is connected due to continuity of $V$.

$\exists \, x \in U$ arbitrarily close to $x = 0 \Rightarrow$ origin on the border of $U$

Consider $\dot{x} = f(x)$ with $x(0) = x_0$. Then

$x_0 \in U \Rightarrow V(x(t)) \geq V(x_0) > V(0) \quad \forall_{t \geq 0}$

Let $M \equiv \max_{x \in \partial B_r} V(x)$

Since $\forall_{x \in U} \; \frac{dV}{dt} > 0 \quad \exists \, t_0$ ST $V(\phi_{t_0}(x_0)) > M$

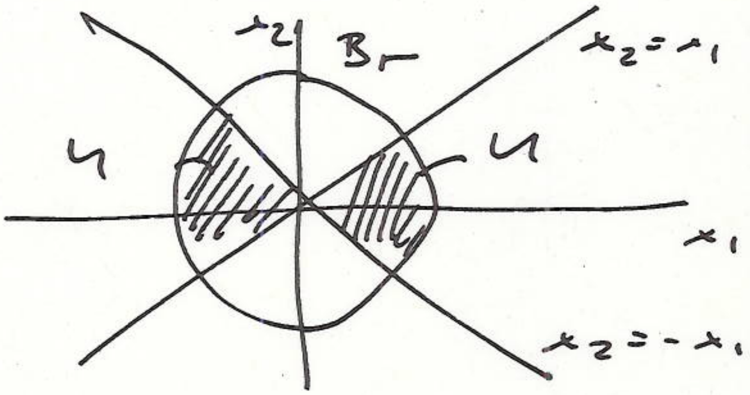
After a finite time $\phi_t(x_0)$ will leave a ball of any radius

**Ex**

$$\begin{cases} \dot{x}_1 = x_1 \\ \dot{x}_2 = -x_2 \end{cases} \Rightarrow V(x) = \frac{1}{2}\left(x_1^2 - x_2^2\right)$$

$$\frac{dV}{dt} = x_1 \dot{x}_1 - x_2 \dot{x}_2 = x_1^2 + x_2^2 > 0$$

$$U = \left\{ (x_1, x_2) : |x_1| < |x_2| \right\}$$




$\Rightarrow$ instability

**Remark**

$V(x)$ need not be sign-definite in the neighbourhood of $x=0$. (less strict requirements than on Lyapunov function in the "stability theorem")

# CHAPTER 3 – HYPERBOLIC THEORY

## 3.1 STABLE & UNSTABLE MANIFOLDS OF DYNAMICAL SYSTEMS

Let $x_* = 0$ be an equilibrium point, i.e. $f(x_*) = 0$ and ~~[illegible]~~ $A = D_x f(x_*)$ be the Jacobian matrix. If $f \in C^1(B_\delta(x_*))$, then $\frac{dx}{dt} = f(x)$ in $x \in B_\delta(x_*)$

can be represented in the "weakly nonlinear" form

$$\dot{x} = Ax + R(x) \quad \text{where} \quad \lim_{\|x\| \to 0} \frac{\|R(x)\|}{\|x\|} = 0$$

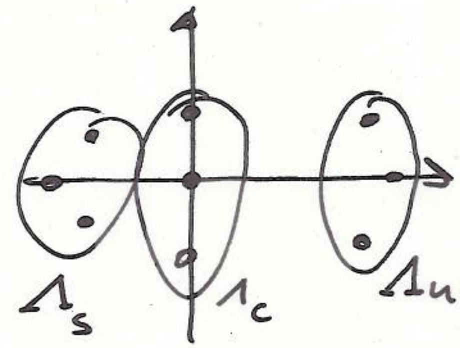
## * Stable and Unstable Manifolds in Linearized Systems

Consider matrix $A$ with eigenvalues $\{\lambda_j\}_{j=1}^{n}$ classified into three groups:

$$\Lambda_s = \{(\lambda_1, \dots, \lambda_m),\ Re\,\lambda_j < 0\}$$

$$\Lambda_u = \{(\lambda_{m+1}, \dots, \lambda_k),\ Re\,\lambda_j > 0\}$$

$$\Lambda_c = \{(\lambda_{k+1}, \dots, \lambda_n),\ Re\,\lambda_j = 0\}$$




Consider sets of eigenvectors (including generalized eigenvectors) associated with eigenvalues in each group:

$$E_s = span\{v_1, \dots, v_m\} \subset \mathbb{R}^n$$

$$E_u = span\{v_{m+1}, \dots, v_k\} \subset \mathbb{R}^n$$

$$E_c = span\{v_{k+1}, \dots, v_n\} \subset \mathbb{R}^n$$

Example

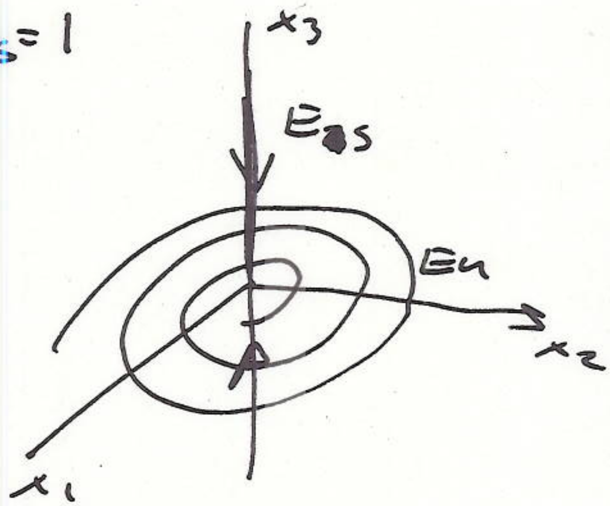
$$\begin{cases} \dot{x}_1 = x_1 + x_2 \\ \dot{x}_2 = -x_1 + x_2 \\ \dot{x}_3 = -x_3 \end{cases} \Rightarrow A = \begin{bmatrix} 1 & 1 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \Rightarrow \begin{array}{l} \lambda_{1,2} = 1 \pm i \\ v_{1,2} = \begin{bmatrix} 1 \\ \pm i \\ 0 \end{bmatrix} \end{array}$$

$$\lambda_3 = -1, \ v_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$E_s = \{x \in \mathbb{R}^3, x_1 = x_2 = 0\}$, $\dim E_s = 1$

$E_u = \{x \in \mathbb{R}^3, x_3 = 0\}$, $\dim E_u = 2$

$E_c = \emptyset$




<u>Lemma</u>

Let $E_0$ be a subspace of $\mathbb{R}^n$ spanned by generalized eigenvectors associated with the eigenvalue $\lambda_0$ of $A$. Then $AE_0 \subset E_0$

<u>Proof</u>

$\forall v \in E_0$, $v = \sum_{j=1}^{k} c_j v_j$. Then $Av = \sum_{j=1}^{k} c_j A v_j$

$$\left.\begin{array}{ll} v_j \text{ - eigenvector} & Av_j = \lambda_j v_j \\ v_j \text{ - generalized eigenvector} & Av_j = \lambda_0 v_j + v_{j-1} \end{array}\right\} \Rightarrow Av_j \in E_0 \Rightarrow AE_0 \subset E_0$$

$\blacksquare$

<u>Def</u>

A subspace $S \subset \mathbb{R}^n$ is <u>invariant</u> w.r.t the phase flow $\phi_t(x)$ if $\forall_{x \in S}$ $\phi_t(x) \subset S$ $\forall_{t \in I \subset \mathbb{R}}$

<u>Ex</u>

A set of critical points of $\dot{x} = f(x)$ is invariant w.r.t $\phi_t(x)$.

<u>Ex</u>

The level sets of the energy function are an invariant set of a <u>conservative</u> dynamical system

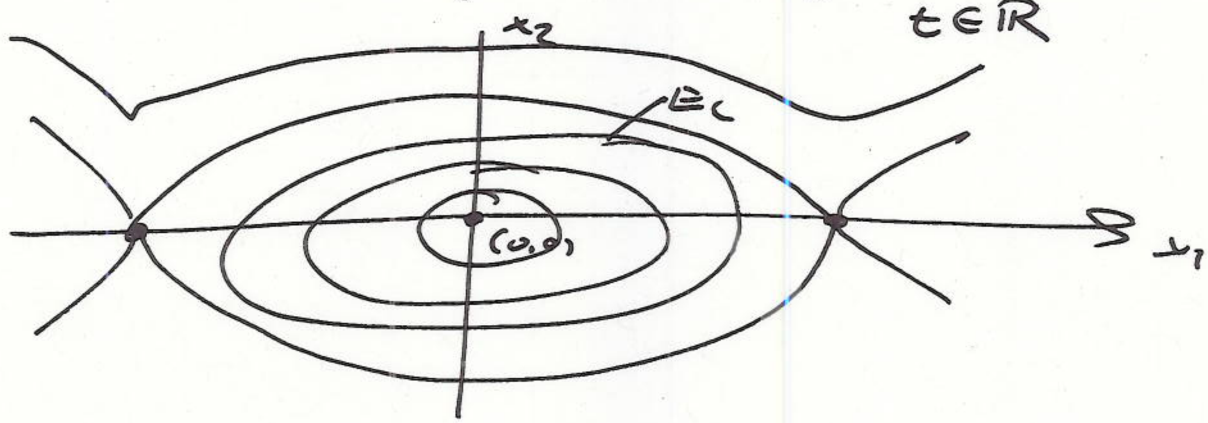
$$\ddot{x} + \sin x = 0 \iff \begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = -\sin x_1 \end{cases}$$

$$E(x_1, x_2) = \frac{1}{2}x_2^2 - \cos x_1 = C$$

$$\frac{dE}{dt} = x_2 \dot{x}_2 + \sin x_1 \dot{x}_1 = \sin x_1(-x_2 + \dot{x}_1) = 0$$

if $x \in S_C = \{x \in \mathbb{R}^2, \frac{1}{2}x_2^2 - \cos x_1 = C\}$ for $t = 0$

then $x \in S_C \ \forall_{t \in \mathbb{R}}$




Thm

The subspaces $E_s, E_u, E_c$ are invariant w.r.t $\phi_t(x) = e^{tA}x \ \forall_{t \in \mathbb{R}}$ and the $\overbrace{\text{decomposition}}^{}$ is unique.

$(\mathbb{R}^n = E_s \oplus E_u \oplus E_c)$

Proof

The generalized eigenvectors of $A$ form a basis for $\mathbb{R}^n$. Furthermore

$\mathbb{R}^n = E_s \oplus E_u \oplus E_c \iff$

$$\forall_{x \in \mathbb{R}^n} \ x = \underbrace{\sum_{j=1}^{m} c_j v_j}_{\in E_s} + \underbrace{\sum_{j=m+1}^{k} c_j v_j}_{\in E_u} + \underbrace{\sum_{j=k+1}^{n} c_j v_j}_{\in E_c} \text{ - unique decomposition}$$

Assume $x_0 \in E_s$, then $x_0 = \sum_{j=1}^{m} c_j v_j$

and

$$x(t) = \phi_t(x_0) = e^{tA} x_0 = \lim_{n\to\infty} \left[I + tA + \ldots + \frac{t^n}{n!} A\right] x_0$$

$$= \sum_{j=1}^{m} c_j \lim_{n\to\infty} \left[I + tA + \ldots + \frac{t^n}{n!} A^n\right] v_j$$

Since $A^k v_j \in E_s \ \forall_{k \geq 1}$, the sum converges absolutely and ~~~~ uniformly. Moreover, since $E_s$ is complete, then $e^{tA} x_0 \in E_s$. Analogously for $E_u$ and $E_c$.

$\blacksquare$

<u>Ex</u>

$$\begin{cases} \dot{x}_1 = 2x_1 + x_2 \\ \dot{x}_2 = 2x_2 \\ \dot{x}_3 = x_3 \end{cases} \Rightarrow A = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

$\lambda_1 = \lambda_2 = 2$, $\lambda_3 = -1$ Thus:

$v_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$, $v_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ $v_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

$E_u = \{x \in \mathbb{R}^3, x_3 = 0\}$

$E_s = \{x \in \mathbb{R}^3, x_1 = x_2 = 0\}$

E.g., if $x_3 = 0$ at $t = 0 \Rightarrow x_3 = 0 \ \forall_{t \geq 0}$

<u>Thm</u>

Let $x_0 \in E_s$. Then, $\exists C > 0$ and $\beta > 0$ ST $e^{tA} x_0 \in E_s$, $\| e^{tA} x_0 \| \leq C e^{-\beta t} \|x_0\|$, $\forall t \geq 0$ and

$$\lim_{t\to\infty} \phi_t(x) = 0$$

Proof

In canonical variables ($y = P^{-1}x$), $PAP^{-1} = J$ has a $k \times k$ block with all eigenvalues $Re(\lambda_j) < 0$. Since $E_s$ is an invariant manifold, the result follows from the previous thm on stable matrices.

Thm

Let $x_0 \in E_u$, $\exists\, C > 0$, $\beta > 0$ st $e^{tA} x_0 \in E_u$, $\| e^{tA} x_0 \| \leq C e^{\beta t} \|x_0\|$, $\forall_{t \leq 0}$ and $\lim_{t \to -\infty} \phi_t(x) = 0$

Proof as before

Because of these properties, the sets $E_s$ and $E_u$ are referred to as the stable and unstable manifolds of the linear system.