HOMEWORK #2

Due: October 28

Instructions:

- The assignment consists of *four* questions, each worth the number of points indicated, for a total value of 15 points.
- All answers should be supported with suitable justifications and/or proofs.
- Submission: electronically (via Email to the instructor) no later than 11:59pm on the due date;
  - Problems #1 and #2: please submit your solutions as a Maple worksheet; make sure that answers to the different tasks in a given problem are clearly identified,
  - Problem #3 and #4): please submit your solutions as a PDF file (no MS Word or PowerPoint files, please!).
- Late submissions and submissions which do not comply with these guidelines will not be accepted.

---

1. You are given the following Cauchy problem:

$$\dot{x} = x^5 + x^3 + 1 \triangleq f(x), \qquad x(0) = 1,$$

for $t \in [0, T)$, where T is some positive real number. Using Maple construct a sequence of Picard's approximations $\{u_k(t)\}_{k=0}^N$ of the solution of this problem

$$\begin{aligned} u_0(t) &= x(0), \\ u_k(t) &= x(0) + \int_0^t f(u_{k-1}(\tau))\, d\tau, \qquad k = 1, \dots, N. \end{aligned}$$

In your calculations use $N = 4$ (note that larger values of N will generate very long outputs which Maple may have difficulty handling). Then

(a) plot all approximations $\{u_k(t)\}_{k=0}^4$ on the interval $[0, 0.1]$,

(b) illustrate (by producing a suitable plot) the uniform Cauchy property of the sequence $\{u_k(t)\}_{k=0}^4$ on the interval $[0, 0.1]$ (you may consider the expressions $|u_k(t) - u_{k-1}(t)|$),

(c) by examining the behavior of $\{u_k(t)\}_{k=0}^4$ estimate with the accuracy of 0.2 the *upper bound* on the interval of t where Picard's iterations converge.

[1 + 2 + 2 = 5 points]

2. Consider the Cauchy problem

$$\dot{x} = \frac{x^3}{x^4+1}, \qquad x(0) = y, \tag{1}$$

for $t \in [0,T)$, where T and y are positive real numbers. Obtain an expression for $\Phi(t) \triangleq \frac{d}{dy}x(t;y)\big|_{y=1}$ in the following two ways:

(a) by constructing a closed–form solution $x(t;y)$ of (1), differentiating it with respect to y and evaluating the resulting expression at $y=1$,

(b) by deriving and solving an initial-value problem satisfied by the quantity $\Phi(t)$.

Plot on one figure the solutions $x(t;1)$, $x(t;3/2)$ of problem (1) corresponding to $y=1$ and $y=3/2$ together with the approximation of the second solution in the form $\tilde{x}(t;3/2) \approx x(t;1) + \Delta y \, \frac{d}{dy}x(t;y)\big|_{y=1}$, where $\Delta y = 1/2$. Generate this plot for $t \in [0,5]$. You are encouraged to use Maple [function `dsolve(...)`] to solve the Cauchy problems in this question.
[2 + 2 + 1 = 5 points]

3. You are given the following Cauchy problem

$$\dot{x} = f(x), \qquad x(t_0) = x_0, \tag{2}$$

in which the function $f : D \to \mathbb{R}^n$ is such that a unique and smooth solution $x(t;t_0,x_0)$ exists on an open interval $\mathcal{I} \ni t_0$. Assuming that this solution is differentiable with respect to t_0, determine the Cauchy problem satisfied by the quantity

$$\Psi(t) \triangleq \frac{\partial x(t;t_0,x_0)}{\partial t_0}.$$

Assume that the initial data x_0 does not depend on t_0.
HINT: it may be useful to consider the integral form of original Cauchy problem (2).
[2 points]

4. You are given the following Cauchy problem

$$\begin{cases} \ddot{y} + y = \sin(t), \\ y(0) = \dot{y}(0) = 0. \end{cases}$$

Obtain the solution to this problem using the formula [for $x(t) \in \mathbb{R}^n$]

$$x(t) = \Phi(t)x_0 + \int_0^t \Phi(t)\,\Phi^{-1}(s)\,b(s)\,ds,$$

where $\Phi(t)$ is the fundamental matrix solution of the Cauchy problem

$$\begin{cases} \dot{\Phi} = A(t)\Phi, \\ \Phi(0) = I. \end{cases}$$

with a suitably defined matrix $A(t)$.
[3 points]