HOMEWORK #3

Due: November 18

Instructions:

- The assignment consists of *four* questions, each worth the number of points indicated, for a total value of 15 points.
- All answers should be supported with suitable justifications and/or proofs.
- Submission: electronically (via Email to the instructor) no later than 11:59pm on the due date;
  - Problem #1: please submit your solutions as a Maple worksheet; make sure that answers to the different tasks in a given problem are clearly identified,
  - Problems #2, #3 and #4: please submit your solutions as a PDF file (no MS Word or PowerPoint files, please!).
- Late submissions and submissions which do not comply with these guidelines will not be accepted.

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1. You are given a systems of linear ODEs with constant coefficients $\dot{\mathbf{x}} = A\mathbf{x}$, where $\mathbf{x}(t) \in \mathbb{R}^5$ and the matrix A is given by

$$A = \begin{bmatrix} -6 & 4 & 6 & -5 & 6 \\ -6 & 4 & 6 & -5 & 6 \\ -7 & 4 & 4 & -3 & 7 \\ -6 & 6 & 0 & -2 & 6 \\ -4 & 4 & 0 & -3 & 4 \end{bmatrix}.$$

   (a) Obtain the fundamental matrix solution $\Phi(t)$ using the real–valued canonical Jordan form of the matrix A.

   (b) Use $\Phi(t)$ to obtain the solution to the Cauchy problem corresponding to the initial condition $\mathbf{x}_0 = [1 \quad -1 \quad 3 \quad 2 \quad 0]^T$; plot the projection of this solution for $t \in [0, 4]$ on

      i. the subspace $S_1 = \operatorname{span}\{\mathbf{u}_1, \mathbf{w}_1\}$, where $\mathbf{u}_1, \mathbf{w}_1 \in \mathbb{R}^5$ and $\mathbf{v}_1 = \mathbf{u}_1 + i\,\mathbf{w}_1$ is the complex eigenvector of A,

      ii. the subspace $S_2 = \operatorname{span}\{\mathbf{u}_2, \mathbf{w}_2\}$, where $\mathbf{u}_2, \mathbf{w}_2 \in \mathbb{R}^5$ and $\mathbf{v}_2 = \mathbf{u}_2 + i\,\mathbf{w}_2$ is the *generalized* complex eigenvector of A.

   [3 + 1 + 1 = 5 points]

2. Compute the exponential e^A of the matrix

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{bmatrix}.$$

   [2 points]

3. A matrix is said to be *semisimple* if the elementary blocks in the Jordan form of the matrix have no ones or identity blocks in the off–diagonal positions. If the system matrix A is *not* semisimple, and all the eigenvalues of A have nonpositive real parts, is it true that if $\dot{\mathbf{x}} = A\mathbf{x}$, then $\exists\, C < \infty$ such that $\|\mathbf{x}(t)\| \le C$, $\forall t \ge 0$? Illustrate your answer with an example.
   [2 points]

4. Classify the equilibrium points (as sinks, sources, or saddles) of the following systems of differential equations:

   (a)

$$\dot{x}_1 = \frac{\sin(x_2)}{x_1 + x_2},$$
$$\dot{x}_2 = -\frac{\sin(x_1)}{x_1 + x_2};$$

   for this system also sketch the phase portrait,

   (b)

$$\dot{x}_1 = x_2 - x_1,$$
$$\dot{x}_2 = k\,x_1 - x_2 - x_1\,x_3,$$
$$\dot{x}_3 = x_1\,x_2 - x_3,$$

   where $k \in \mathbb{R}\backslash\{1\}$ is a parameter; note that the locus and type of the critical points in this problem may depend on the value of k.

   [3 + 3 = 6 points]