MATH 741

HOMEWORK ASSIGNMENT #3 - SOLUTIONS

(2) $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{bmatrix}$, $e^A = ?$

$$e^A = P e^J P^{-1}$$

Matrix A is already in Jordan form:

$A = J = \begin{bmatrix} J_1 & 0 \\ 0 & J_2 \end{bmatrix}$, $J_1 = 1$, $J_2 = \begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix}$

so that $P = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = P^{-1}$ and $e^J = \begin{bmatrix} e^{J_1} & 0 \\ 0 & e^{J_2} \end{bmatrix}$.

We have:

$$e^{J_1} = e^1$$

By ~~Corollary~~ Corollary 4 in Section 1.3 of [Perko]

$$Q = \begin{bmatrix} a & b \\ 0 & a \end{bmatrix} \Rightarrow e^Q = e^a \begin{bmatrix} 1 & b \\ 0 & 1 \end{bmatrix}$$

$$e^{J_2} = e^2 \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$$

and finally

$$e^A = \begin{bmatrix} e & 0 & 0 \\ 0 & e^2 & e^2 \\ 0 & 0 & e^2 \end{bmatrix}$$

③ No, in fact one may have

$$\lim_{t \to \infty} \|x(t)\| = \infty$$

cf. Example 4 in Section 1.7 in [Perko].

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④ a)

$$\dot{x}_1 = \frac{\sin(x_2)}{x_1 + x_2} = f_1(x_1, x_2)$$

$$\dot{x}_2 = - \frac{\sin(x_1)}{x_1 + x_2} = f_2(x_1, x_2)$$

Critical points:

at $x_1 + x_2 = 0$, f_1, f_2 not defined

$f_1(x_1, x_2) = 0 \iff \sin(x_2) = 0 \iff x_2 = n\pi, \, n \in \mathbb{Z}$

$f_2(x_1, x_2) = 0 \iff \sin(x_1) = 0 \iff x_1 = m\pi, \, m \in \mathbb{Z}$

Thus

$$(x_1, x_2) = (m\pi, n\pi), \quad m + n \neq 0$$

Jacobian matrix:

$$A = D_x f = \begin{bmatrix} \frac{\partial f_1}{\partial x_1}(x_1, x_2) & \frac{\partial f_1}{\partial x_2}(x_1, x_2) \\ \frac{\partial f_2}{\partial x_1}(x_1, x_2) & \frac{\partial f_2}{\partial x_2}(x_1, x_2) \end{bmatrix} =$$

$$
= \begin{bmatrix} -\dfrac{\sin(x_2)}{(x_1+x_2)^2} & \dfrac{-\sin(x_2)+(x_1+x_2)\cos(x_2)}{(x_1+x_2)^2} \\[2ex] \dfrac{-(x_1+x_2)\cos(x_1)+\sin(x_1)}{(x_1+x_2)^2} & \dfrac{\sin(x_1)}{(x_1+x_2)^2} \end{bmatrix}
$$

Evaluating the Jacobian at the critical points:

$$
A\Big|_{\substack{(m\pi,\, n\pi) \\ m+n\neq 0}} = \begin{bmatrix} 0 & \dfrac{(-1)^n}{(n+m)\pi} \\[2ex] -\dfrac{(-1)^m}{(n+m)\pi} & 0 \end{bmatrix}
$$

Eigenvalues:

$$
\lambda_\pm = \pm \frac{\sqrt{(-1)^{n+m+1}}}{(n+m)\pi}
$$

$(n+m+1)$ — even

$$
\lambda_\pm = \pm \frac{1}{(n+m)\pi} \quad \text{— saddle point}
$$

$(n+m+1)$ — odd

$$
\lambda_\pm = \pm \frac{i}{(n+m)\pi} \quad \text{— center point}
$$

Eigenvalues and eigenvectors at saddle point

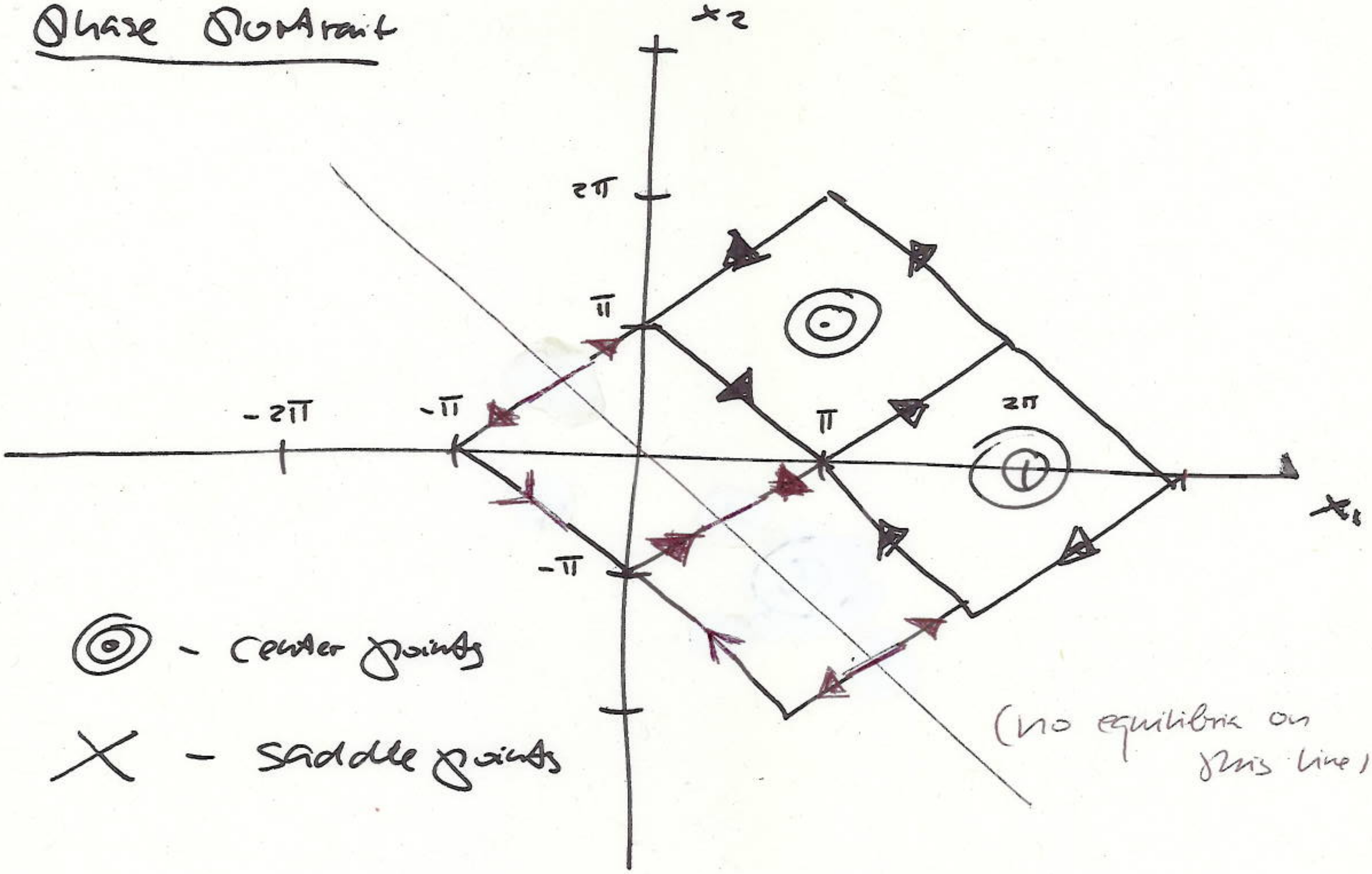
$(m,n) = (1,0)$

$$
\lambda_\pm = \pm\frac{1}{\pi}, \quad v_+ = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad v_- = \begin{bmatrix} -1 \\ 1 \end{bmatrix}
$$

$(m,n) = (0,1)$

$$
\lambda_\pm = \pm\frac{1}{\pi}, \quad v_+ = \begin{bmatrix} -1 \\ 1 \end{bmatrix} \quad v_- = \begin{bmatrix} 1 \\ 1 \end{bmatrix}
$$

Phase Portrait



(4b)

$$\begin{cases} \dot{x}_1 = x_2 - x_1 \\ \dot{x}_2 = k x_1 - x_2 - x_1 x_3 \\ \dot{x}_3 = x_1 x_2 - x_3 \end{cases} \implies \begin{cases} x_1 = x_2 \\ x_1 (k - 1 - x_1^2) = 0 \\ x_1^2 = x_3 \end{cases}$$

Equilibrium points:

if $k \leq 1$

~~xxxx~~ $(0, 0, 0)$

if $k > 1$

$(0, 0, 0)$

$(\sqrt{k-1}, \sqrt{k-1}, k-1)$

$(-\sqrt{k-1}, -\sqrt{k-1}, k-1)$

Jacobian matrix

$$D_x f(x) = \begin{bmatrix} -1 & 1 & 0 \\ k - x_3 & -1 & -x_1 \\ x_2 & x_1 & -1 \end{bmatrix}$$

Evaluating $D_x f(x)$ at the <u>critical point $(0,0,0)$</u>

$$D_x f((0,0,0)) = \begin{bmatrix} -1 & 1 & 0 \\ k & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

Eigenvalues:
$$\begin{cases} \lambda_1 = -1 \\ \lambda_2 = -1 - \sqrt{k} \\ \lambda_3 = -1 + \sqrt{k} \end{cases} \Rightarrow$$

If $k \in (1, \infty)$

$\lambda_1, \lambda_2 < 0$

$\lambda_3 > 0$

<u>SADDLE POINT</u>

If $k \in (-\infty, 1)$

$Re(\lambda_1), Re(\lambda_2), Re(\lambda_3) < 0$

SINK POINT

Evaluating $D_x f(x)$ at the critical points $(\pm\sqrt{k-1}, \pm\sqrt{k-1}, k-1)$

$$D_x f(\pm\sqrt{k-1}, \pm\sqrt{k-1}, k-1) =$$
$$= \begin{bmatrix} -1 & 1 & 0 \\ 1 & -1 & \mp\sqrt{k-1} \\ \pm\sqrt{k-1} & \pm\sqrt{k-1} & -1 \end{bmatrix}$$

Eigenvalues:

$$\lambda_1 = -2, \quad \lambda_{2,3} = \frac{-1 \pm \sqrt{5-4k}}{2}$$

If $k \in (1, \frac{5}{4}]$

$\lambda_1, \lambda_2, \lambda_3$ - all real

$\lambda_1, \lambda_2, \lambda_3 < 0$

If $k \in (\frac{5}{4}, \infty)$

λ_2, λ_3 - complex

$Re(\lambda_2) = Re(\lambda_3) < 0$

~~Eigenvalues~~:

<u>SINK POINT</u>