HOMEWORK #4

Due: December 2

Instructions:

- The assignment consists of *four* questions, each worth the number of points indicated, for a total value of 15 points.
- All answers should be supported with suitable justifications and/or proofs.
- Submission: electronically (via Email to the instructor) no later than 11:59pm on the due date;
  - Problem #5: please submit your solutions as a Maple worksheet; make sure that answers to the different tasks in a given problem are clearly identified,
  - Problems #1, #2, #3 and #4: please submit your solutions as a PDF file (no MS Word or PowerPoint files, please!).
- Late submissions and submissions which do not comply with these guidelines will not be accepted.

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1. Let $g : \mathbb{R}^n \to \mathbb{R}^n$. Show that $g(x)$ is the gradient of a scalar function $V : \mathbb{R}^n \to \mathbb{R}$, $V \in C^2(\mathbb{R}^n)$, if and only if

$$\frac{\partial g_i}{\partial x_j} = \frac{\partial g_j}{\partial x_i}, \qquad \forall i, j = 1, \dots, n.$$

   [1 point]

2. Consider the following two systems of nonlinear ODEs

   (a)
$$\begin{cases} \dot{x}_1 = x_2, \\ \dot{x}_2 = -h(x_1) - a\, x_2, \end{cases}$$

   (b)
$$\begin{cases} \dot{x}_1 = x_2, \\ \dot{x}_2 = -h(x_1) - a\, x_1, \end{cases}$$

   where $a > 0$ whereas $h : \mathbb{R} \to \mathbb{R}$ is a locally Lipschitz function such that $h(0) = 0$ and $y\, h(y) > 0$ $\forall y \in \mathbb{R}$. Determine the stability of the origin $(x_1, x_2) = (0, 0)$ in the two cases.
   [2 + 2 = 4 points]

3. Suppose $Q \in \mathbb{R}^{n \times n}$ is a symmetric, positive–definite matrix. Prove that if the origin $x = 0$ of the linear dynamical system with constant coefficients $\dot{x} = Ax$, where $A \in \mathbb{R}^{n \times n}$, is asymptotically stable, then there exists a symmetric, positive–definite solution $P \in \mathbb{R}^{n \times n}$ of the *Lyapunov equation*

$$PA + A^T P = -Q. \tag{1}$$

   [3 points]

4. Consider the following system of nonlinear ODEs

$$\begin{cases} \dot{x}_1 = x_1 + g_1(x), \\ \dot{x}_2 = -x_2 + g_2(x), \end{cases}$$

where $x \triangleq [x_1,\ x_2]^T$ and the functions $g_1, g_2 \ :\ D \to \mathbb{R}$, where $D \subset \mathbb{R}^2$ is some neighbourhood of the origin, are locally Lipschitz and satisfy the estimates

$$|g_1(x)| \le k\,\|x\|_2^2, \qquad |g_2(x)| \le k\,\|x\|_2^2, \quad \forall x \in D,$$

in which $k > 0$. Prove that the origin $(x_1, x_2) = (0, 0)$ is an unstable equilibrium point.
[3 points]

5. You are given the following system of ODEs

$$\begin{cases} \dot{x} = -\dfrac{x}{2} + 2\,x^2\,y, \\ \dot{y} = x - y - x^3, \end{cases} \tag{2}$$

whose origin $(x, y) = (0, 0)$ is asymptotically stable. Use Maple to carry out the following tasks

(a) using linearization of system (2) around the origin together with equation (1) from Problem #3 and the assumption that Q is an identity matrix construct a Lyapunov function $V(x, y)$ for system (2) in the neighbourhood of the origin,

(b) use this Lyapunov function to estimate the boundaries of the stability region $S \ni (0, 0)$ of system (2); in addition to obtaining an expression $y = y(x)$ for the boundary of S, produce a plot illustrating S on the plane (x, y).

[3 + 1 = 4 points]