MATH 741 – HOMEWORK ASSIGNMENT #4
SOLUTIONS

1) $V:\mathbb{R}^n \to \mathbb{R}$, $V \in C^2(\mathbb{R}^n)$, $\nabla V = \left(\frac{\partial V}{\partial x_1}, \dots, \frac{\partial V}{\partial x_n}\right)$

* in the "forward" direction

$$\frac{\partial g_i}{\partial x_j} = \frac{\partial^2 V}{\partial x_j \partial x_i} = \frac{\partial^2 V}{\partial x_i \partial x_j} = \frac{\partial g_j}{\partial x_i} \quad \forall_{i,j}$$

— partial derivatives can be swapped because $V \in C^2(\mathbb{R}^n)$

* in the "reverse" direction

$$\frac{\partial g_j}{\partial x_i} = \frac{\partial g_i}{\partial x_j} \Rightarrow \int \frac{\partial g_j}{\partial x_i} dx_i = g_j = \int \frac{\partial g_i}{\partial x_j} dx_i = \frac{\partial}{\partial x_j} \int g_i dx_i \quad (*)$$

Defining $V(x_1, \dots, x_n) = \int g_i dx_i$ such that $g_i = \frac{\partial V}{\partial x_i}$

we also have from (*) $g_j = \frac{\partial V}{\partial x_j}$, $\forall_j$.

Thus,

$[g_1(x), \dots, g_n(x)] = \nabla V$ iff $\frac{\partial g_i}{\partial x_j} = \frac{\partial g_j}{\partial x_i}$, $\forall_{i,j = 1, \dots, n}$

2a)
$$\begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = -h(x_1) - a x_2 \end{cases} , \alpha > 0$$

$h(\cdot)$ – Locally Lipschitz and $h(0) = 0$, $y h(y) > 0$ $\forall_{y \neq 0}$

Consider Lyapunov function

$$V(x_1, x_2) = \frac{x_2^2}{2} + \int_0^{x_1} h(y) dy$$

Note that in view of the assumptions on $h(y)$

$$\forall_{z \neq 0} \int_0^z h(y) dy > 0$$

hence $V(x_1, x_2)$ satisfies P1 and P2. As regards P3

$$\frac{dV}{dt} = x_2 \dot{x}_2 + h(x_1)\dot{x}_1 = h(x_1)x_2 + x_2\left[-h(x_1) - a x_2\right]$$

$$= -a x_2^2 \leq 0 \quad \forall_{x_1, x_2 \neq 0}$$

Therefore, the origin is stable. See additional comment further below on page 3.

(2b)
$$\begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = -h(x_1) - a x_1 \end{cases}$$

Consider now a different Lyapunov function

$$V(x_1, x_2) = a\frac{x_1^2}{2} + \frac{x_2^2}{2} + \int_0^{x_1} h(y)\,dy$$

Properties P1 and P2 are satisfied

Property 3:

$$\frac{dV}{dt} = \left[h(x_1) + a x_1\right]\dot{x}_1 + x_2 \dot{x}_2$$

$$= \left[h(x_1) + a x_1\right]x_2 + x_2\left[-h(x_1) - a x_1\right] = 0$$

Hence, the origin is <u>stable</u>, although not necessarily <u>asymptotically stable</u>

(2a) cont

Alternatively, one could analyze the linearization

$$A = \begin{bmatrix} 0 & 1 \\ -h'(0) & -a \end{bmatrix}$$

The eigenvalues $\lambda_{1/2} = -\frac{a}{2} \pm \frac{1}{2}\sqrt{a^2 - 4h'(0)}$

hence $Re(\lambda_{1/2}) < 0 \Rightarrow$ asymptotic stability

In the system from (2b) the origin is not a hyperbolic point

(3) If A - asymptotically stable, then all eigenvalues are such that $Re(\lambda_j) < 0$

Consider the ~~[illegible]~~ function

$V(x) = x^T P x$, where $P \in \mathbb{R}^{n \times n}$

(1) ~~[illegible]~~ $V(0) = 0$

(2) if P - ~~[illegible]~~ positive definite

$V(x) = x^T P x > 0 \quad \forall x \in \mathbb{R}^n \setminus \{0\}$

(3) $\frac{dV}{dt} = x^T P \dot{x} + \dot{x}^T P x =$

$= x^T P A x + x^T A^T P x = x^T (PA + A^T P) x$

If P satisfies the Lyapunov equation

$$PA + A^T P = -Q \quad \text{, then}$$

$$\frac{dV}{dt} = -x^T Q x \leq 0 \quad \forall x \in \mathbb{R}^n$$

Furthermore, since Q is symmetric, the Lyapunov equation must be invariant w.r.t transposition

$$-Q^T = -Q = (PA)^T + (A^T P)^T$$

$$\left.\begin{aligned} &= A^T P^T + P^T A \\ &= A^T P + PA \end{aligned}\right\} \text{true iff } P \text{ is } \underline{\text{symmetric}}$$

~~Thus, if V(x) is a Lyapunov function~~

Thus, if $V(x) = x^T P x$ is a Lyapunov function for $\dot{x} = Ax$, $P \in \mathbb{R}^{n \times n}$ must be a symmetric, positive-definite solution of the Lyapunov equation ■

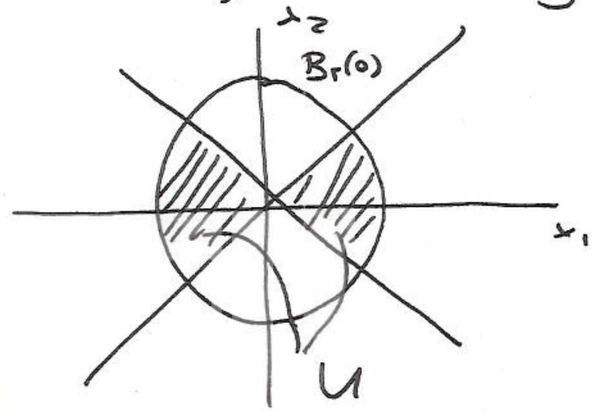
4. We shall use $\|\cdot\| \equiv \|\cdot\|_2$

$$\left.\begin{array}{l} \forall_{x\in D} \; |g_1(x)| \le k\|x\|^2 \\ \qquad |g_2(x)| \le k\|x\|^2 \end{array}\right\} \Rightarrow g_1(0) = g_2(0) = 0$$

Thus, the origin $(0,0)$ is an equilibrium point.

Consider $V(x) = \frac{1}{2}(x_1^2 - x_2^2)$
and the set $U = \{(x_1, x_2) \in B_r(0),\ |x_1| > |x_2|\}$

How to choose r?



Start with

$$\frac{dV}{dt} = x_1 \dot{x}_1 - x_2 \dot{x}_2$$

$$= x_1^2 + x_2^2 + \underbrace{x_1 g_1(x) - x_2 g_2(x)}$$

now estimate these terms

$$|x_1 g(x) - x_2 g_2(x)| \le |x_1|\,|g_1(x)| + |x_2|\,|g_2(x)|$$
$$\le (|x_1| + |x_2|)\, k\|x\|^2 \le \underline{2k\|x\|^3}$$

Then

$$\frac{dV}{dt} = x_1^2 + x_2^2 + x_1 g_1(x) - x_2 g_2(x)$$
$$\ge \|x\|^2 - 2k\|x\|^3 = \|x\|^2 \underbrace{(1 - 2k\|x\|)}_{\text{need to choose } r \text{ in } B_r(0) \text{ to make this positive}}$$

Set $r = \frac{1}{2k}$

and finally $\frac{dV}{dt} > 0 \ \forall_{x \in U}$

and the origin $(0,0)$ is ~~stable~~ an unstable critical point by the Lyapunov instability theorem.