The fundamental matrix $\phi(t) = e^{tA}$ satisfies

$$\begin{cases} \dot{\phi} = A\phi \\ \phi(0) = I \end{cases}$$

CASE II — Multiple real eigenvalues

Even if the matrix A is not diagonalizable (due to multiple or complex eigenvalues), it can be represented in a canonical block-diagonal form (Jordan form)

Definition

Let λ_0 be an eigenvalue of A ST

$\det(A - \lambda I) = (\lambda - \lambda_0)^k D_k(\lambda)$, where

$D_k(\lambda_0) \neq 0$ and $\dim \mathrm{Null}(A - \lambda I) = m \leq k$

Then, k is the algebraic multiplicity of λ_0, and $m \leq k$ is the geometric multiplicity of λ_0.

Ex

$$\begin{bmatrix} \lambda_0 & 0 \\ 0 & \lambda_0 \end{bmatrix} \qquad \begin{bmatrix} \lambda_0 & 1 \\ 0 & \lambda_0 \end{bmatrix}$$

$k = m = 2$ $\{\bar{e}_1, \bar{e}_2\}$; $k = 2,\ m = 1$ $\{\bar{e}_1\}$

$$\begin{bmatrix} \lambda_0 & 0 & 0 \\ 0 & \lambda_0 & 0 \\ 0 & 0 & \lambda_0 \end{bmatrix} \qquad \begin{bmatrix} \lambda_0 & 1 & 0 \\ 0 & \lambda_0 & 0 \\ 0 & 0 & \lambda_0 \end{bmatrix} \qquad \begin{bmatrix} \lambda_0 & 1 & 0 \\ 0 & \lambda_0 & 1 \\ 0 & 0 & \lambda_0 \end{bmatrix}$$

$k = m = 3$; $k = 3,\ m = 2$; $k = 3,\ m = 1$

Def

Let k be the algebraic multiplicity of λ_0 and $m=1$. Then, the set $\{v_0, \dots, v_{k-1}\}$ defined as

$$\begin{cases} (A-\lambda_0 I)v_0 = 0 \\ (A-\lambda_0 I)v_1 = v_0 \\ \dots\dots \\ (A-\lambda_0 I)v_{k-1} = v_{k-2} \end{cases}$$

is called the <u>chain of generalized eigenvectors</u>

Remark

$$(A-\lambda_0 I)^{j+1} v_j = 0 \quad , j = 0, \dots, k-1$$

Lemma (Jordan)

Let A have only real eigenvalues. The set of all generalized eigenvectors is linearly independent. If $P = [v_0, \dots, v_{n-1}]$, then

$$P^{-1}AP = J = \begin{bmatrix} J_1 & 0 & \dots & 0 \\ 0 & J_2 & \dots & 0 \\ \vdots & & & \\ 0 & 0 & \dots & J_s \end{bmatrix} \quad \text{where}$$

$$J_j = \begin{bmatrix} \lambda_j & 1 & \dots & 0 \\ 0 & \lambda_j & \dots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \dots & \lambda_j \end{bmatrix}$$ is a $k_j \times k_j$ Jordan block and $n = k_1 + k_2 + \dots + k_s$

Proof

Linear independence – see textbook on <u>algebra</u>

Consider generalized eigenvectors for one eigenvalue λ_0 only

$$\lambda_0 \to \{v_0, \dots, v_{k-1}\}$$

$$P^{-1}AP = P^{-1}A[v_0, v_1, \dots, v_{k-1}] = P^{-1}[\lambda_0 v_0, \lambda_0 v_1 + v_0, \dots, \lambda_0 v_k + v_{k-1}]$$

$$= P^{-1}P\begin{bmatrix} \lambda_0 & 1 & \dots & 0 \\ 0 & \lambda_0 & \dots & 0 \\ \vdots & \vdots & & \\ 0 & 0 & \dots & \lambda_0 \end{bmatrix} = J$$

properties of generalized eigenvectors

Corollary

$$e^{tA} = P^{-1}E(t)P, \quad E(t) = e^{tJ} = \begin{bmatrix} E_1(t) & 0 & \dots & 0 \\ 0 & E_2(t) & \dots & 0 \\ \vdots & \vdots & & \\ 0 & 0 & \dots & E_s(t) \end{bmatrix}$$

where $E_j(t) = e^{tJ_j}$, $j = 1, \dots, s$

Definition

A $k \times k$ matrix N_k is call nilpotent of order p if $N_k, N_k^2, \dots, N_k^{p-1} \neq 0$, and $N_k^p = 0$

Lemma

$$J_j = \lambda_j I_{k_j} + N_{k_j}$$

such that

$$E_j(t) = e^{t\lambda_j}\left[I_{k_j} + tN_{k_j} + \tfrac{1}{2}t^2 N_{k_j}^2 + \dots + \tfrac{1}{(k_j-1)!}t^{k_j-1}N_{k_j}^{k_j-1}\right]$$

Proof

In Jordan canonical form

$$N_{k_j} = \text{shift matrix} \begin{bmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 1 \\ 0 & 0 & 0 & \dots & 0 \end{bmatrix}$$

nilpotent of degree N_{k_j}

$$e^{t\lambda_j I_{k_j} + tN_{k_j}} = e^{t\lambda_j I_{k_j}} e^{tN_{k_j}} = e^{t\lambda_j} I_{k_j} e^{tN_{k_j}}$$

I_{k_j} and N_{k_j} commute

$$= e^{t\lambda_j}\left[I_{k_j} + tN_{k_j} + \dots + \frac{1}{(k_j-1)!} t^{k_j-1} N_{k_j}^{k_j-1}\right]$$

$\blacksquare$

Corollary

The solution $x(t)$ of $\begin{cases} \dot{x} = Ax \\ x(0) = x_0 \end{cases}$ is given by

$$x(t) = P \operatorname{diag}\{e^{t\lambda_j}\} P^{-1} \left[I + Nt + \dots + \frac{1}{(k-1)!} t^{k-1} N^{k-1}\right] x_0$$

where $N = A - S = A - P\operatorname{diag}\{e^{t\lambda_j}\}P^{-1}$

CASE III – Complex Eigenvalues

There are two ways to deal with complex eigenvalues:

* Complexify the matrices P and J
* Modify the canonical Jordan representation

Ex

$$A = \begin{bmatrix} a & b \\ -b & a \end{bmatrix} \Rightarrow \begin{matrix} \lambda_1 = a + ib \\ \lambda_2 = a - ib = \bar{\lambda}_1 \end{matrix} \Rightarrow \begin{matrix} Av_1 = \lambda_1 v_1 \\ Av_2 = \lambda_2 v_2 \end{matrix} \Rightarrow \begin{cases} v_1 = \begin{bmatrix} 1 \\ i \end{bmatrix} \\ v_2 = \begin{bmatrix} 1 \\ -i \end{bmatrix} \end{cases}$$

Then

$$P = [v_1\ v_2] = \begin{bmatrix} 1 & 1 \\ i & -i \end{bmatrix}, \quad P^{-1} = \frac{1}{2}\begin{bmatrix} 1 & -i \\ 1 & i \end{bmatrix}$$

$$P^{-1}AP = D = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}$$ – everything is complex.

When constructing the solution, we need to ensure that it is ~~complete~~ real-valued:

$$x(t) = c_1 v_1 e^{\lambda_1 t} + \overline{c_1}\, \overline{v_1}\, e^{\overline{\lambda_1} t}$$

On the other hand, one can work with real eigenvectors and Jordan blocks

$$\begin{cases} v_1 = u + iw \\ v_2 = u - iw \end{cases}, \quad u, w \in \mathbb{R}^2$$

So that $Av_1 = \lambda_1 v_1 \Rightarrow \begin{cases} Au = au - bw \\ Av = aw + bu \end{cases}$

Let now $P = [u, w]$

Then $P^{-1}AP = P^{-1}[au - bw, aw + bu]$

$$= P^{-1}P \begin{bmatrix} a & b \\ -b & a \end{bmatrix} = \begin{bmatrix} a & b \\ -b & a \end{bmatrix} = aI + b\sigma, \quad \sigma = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \sigma^2 = -I$$

Therefore

$$E(t) = e^{tJ} = \left| \begin{matrix} I \text{ and } \sigma \\ \text{commute} \end{matrix} \right| = e^{aIt} \cdot e^{b\sigma_1 t}$$

$$= e^{at} \underbrace{\left[I + \sigma_1 tb + \frac{1}{2!} t^2 b^2 I - \frac{1}{3!} t^3 b^3 \sigma_1 + \dots \right]}_{\text{recognizing series expansions}}$$

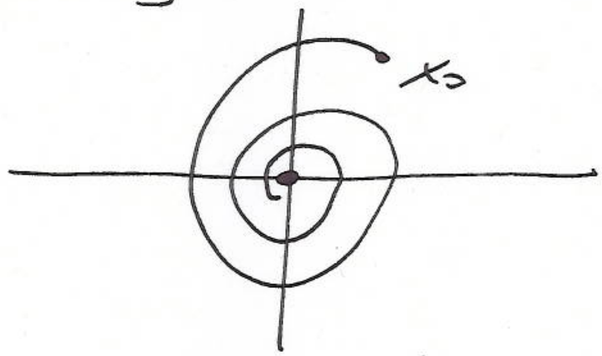
$$= e^{at}\left[\cos bt\, I + \sin bt\, \sigma_1 \right] = e^{at} \underbrace{\begin{bmatrix} \cos bt & \sin bt \\ -\sin bt & \cos bt \end{bmatrix}}_{R(bt)}$$

$$= e^{at} R(bt)$$

$R(bt)$ orthogonal matrix of rotation

Ex (in the normal coordinates)

$$x(t) = E(t)x_0 = \underbrace{e^{at}}_{\text{growth or decay}} \underbrace{R(bt)}_{\text{rotation}} x_0 = c_1 e^{at}\begin{bmatrix}\cos bt\\ -\sin bt\end{bmatrix} + c_2 e^{at}\begin{bmatrix}\sin bt\\ \cos bt\end{bmatrix}$$



if $a<0$, $(0,0)$ is a stable focus

Thm (no proof)

Let all eigenvalues $\lambda_k = a_k + i b_k$, $k=1,\dots,n$ of A be complex and distinct. Let $v_k = u_k + i w_k$ be the corresponding eigenvectors. Then, $P = [u_1, w_1, \dots, u_n, w_n]$ is invertible and

$$P^{-1}AP = J = \begin{bmatrix} J_1 & & 0 \\ & \ddots & \\ 0 & & J_n\end{bmatrix}, \text{ where } J_k = \begin{bmatrix} a_k & b_k \\ -b_k & a_k\end{bmatrix}$$

Remark

Multiple complex eigenvalues can be treated similarly to multiple real eigenvalues by using chains of generalized eigenvectors.

General Jordan block diagonalization in $\mathbb{R}^n$

$$J = \begin{bmatrix} J_1 & & 0 \\ & \ddots & \\ 0 & & J_s\end{bmatrix} \text{ where } J_j = \begin{bmatrix} \lambda_j & 1 & 0 & \dots & 0 \\ 0 & \lambda_j & 1 & 0 & \dots \\ \vdots & & & \ddots & \\ 0 & 0 & 0 & \dots & \lambda_j\end{bmatrix}$$

or

$$\mathcal{J}_j = \begin{bmatrix} a_j & b_j & 1 & 0 & \cdots \\ -b_j & a_j & 0 & 1 & \cdots \\ 0 & 0 & a_j & b_j & \cdots \\ 0 & 0 & -b_j & a_j & \\ \vdots & \vdots & \vdots & \vdots & \end{bmatrix}, \quad j = 1, \dots, s$$

~~Example~~ Corollary

Each component of ~~a~~ the solution $x(t)$ of Cauchy problem $\begin{cases} \dot{x} = Ax \\ x(0) = x_0 \end{cases}$ is a linear combination of functions of the form

$$t^k e^{at} \cos bt \quad \text{or} \quad t^k e^{at} \sin bt,$$

where $0 \le k \le n-1$ and $\lambda = a + bi$ is an eigenvalue of A.

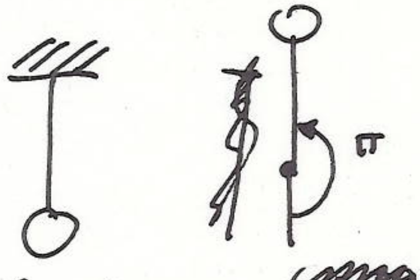
2.3 CRITICAL POINTS & LINEARIZED STABILITY

Lyapunov's First Method – analysis of stability of a nonlinear system based on its linearization.

Among other solutions of ODEs, constant functions are important, because they are critical points

Ex (Pendulum)

$$\ddot{x} + \sin x = 0 \Rightarrow \frac{d}{dt}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} x_2 \\ -\sin x_1 \end{pmatrix}$$



$(0,0)$ $(\pi,0)$

Equilibrium points

$$(x_1, x_2) = \{(0,0), (\pi,0), (-\pi,0)\}$$

+ periodic copies

Ex (Lotka – Volterra system)

$$\begin{cases} \dot{x}_1 = -a x_1 + b x_1 x_2 \\ \dot{x}_2 = \quad\quad - b x_1 x_2 \end{cases} \Rightarrow \begin{cases} x_1 = 0 \\ x_2 \in \mathbb{R} \end{cases}$$

$a, b > 0$

Critical points are classified into <u>isolated points</u> (first example) and <u>nonisolated points</u> (second example).

<u>Def</u>

Critical point $x_* \in D$ is said to be isolated if $B_r(x_*) \in D$ contains no other critical points for some $r > 0$.

<u>Lemma</u>

Let f be C^1 and $D_x f(x_*)$ be the Jacobian matrix for $\dot{x} = f(x)$ evaluated at $x = x_*$. The point x_* is an ~~[illegible]~~ <u>isolated</u> critical point of $\dot{x} = f(x)$ if $f(x_*) = 0$ and $D_x f(x_*)$ is nonsingular.

Proof (by contradiction)

Let $x_*(\varepsilon)$, depending on $\varepsilon \in \mathbb{R}^m$, $m \geq 1$, be a C^1 family of critical points. Then

$$\underbrace{f(x_*(\varepsilon))}_{0} = \underbrace{f(x_*(0))}_{0} + D_x f(x_*) \cdot D_\varepsilon x_*(0) \cdot \varepsilon + R_\varepsilon(x) = 0$$

such that

$$\lim_{\varepsilon \to 0} \frac{\| R_\varepsilon(x^*) \|}{\| \varepsilon \|} = 0$$

Divide into ε and take the limit $\varepsilon \to 0$

$$D_x f(x_*) \cdot D_\varepsilon x_*(0) = 0 \Rightarrow D_x f(x_*)$$

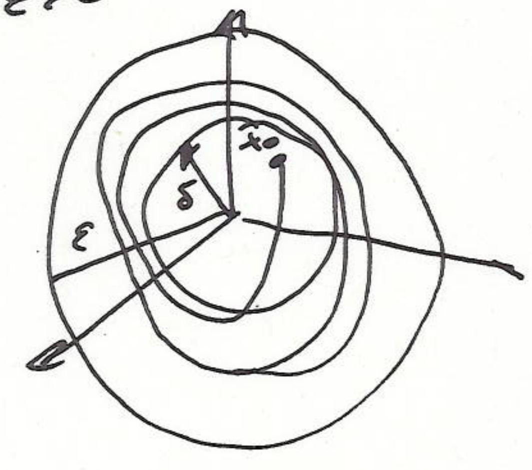
must be singular (necessary condition)

Remark

We will often assume that the critical point is at the origin, since if $x_* \neq 0$, then $w = x - x_*$ satisfies $\frac{dw}{dt} = f(x_* + w) = \tilde{f}(w)$ ST $f(0) = 0$.

Def

An equilibrium point x_* is stable if $\forall \varepsilon > 0$ $\forall_{\varepsilon > 0} \ \exists\, \delta(\varepsilon) > 0$ ST $\forall_{x_0 \in B_\delta(x_*)}$ we have $x(t) = \phi_t(x_0) \in B_\varepsilon(x_*)$



If it is not stable, it is unstable

In this case

$\forall_{\varepsilon > 0} \ \forall_{x_0 \in B_\varepsilon(x_*)} \ \exists\, t_0 > 0$ ST $x(t_0) \notin B_\varepsilon(x_*)$