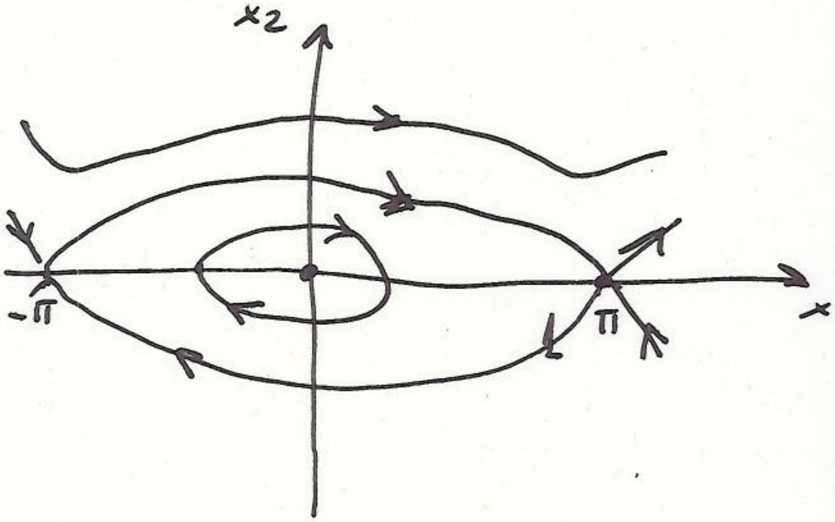
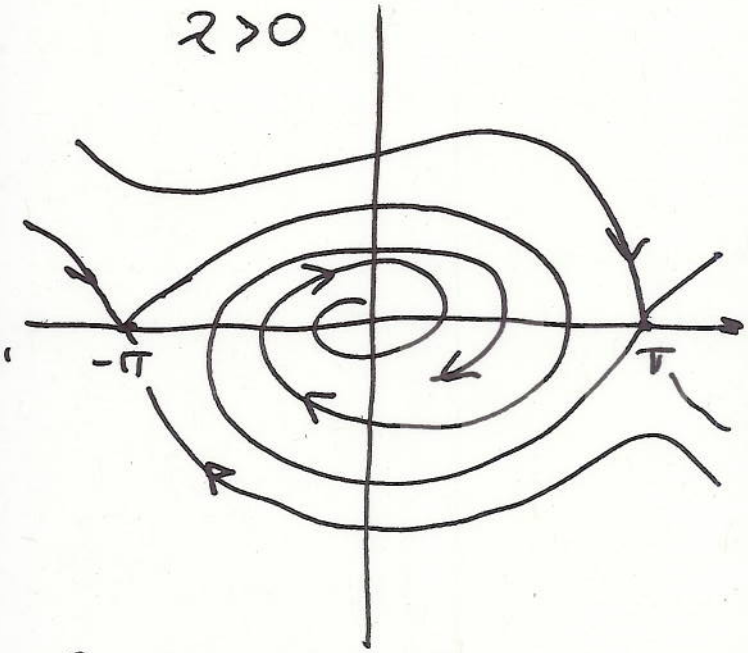
<u>Ex</u> (damped pendulum) $\ddot{x} + \lambda \dot{x} + \sin x = 0$

$\lambda = 0$



$(0,0)$ – stable (center)
$(\pi,0)$, $(-\pi,0)$ – unstable (saddle)

$\lambda > 0$



$(0,0)$ – asymptotically stable (sink)
$(\pi,0)$, $(-\pi,0)$ – unstable (saddle)

<u>Def</u>
The equilibrium point x_* is <u>asymptotically stable</u> if it is stable and there exists a $\delta > 0$ ST
$$\forall_{x_0 \in B_\delta(x_*)} \quad \lim_{t \to \infty} \phi_t(x_0) = x_*$$
The set of all $x_0 \in \mathbb{R}^n$ for which $\lim_{t\to\infty} \phi_t(x_0) = x_*$ is called the <u>domain of attraction</u> of the ~~equi~~ equilibrium point x_*.

<u>Linearization and Stability of Critical Points</u>

From Taylor's theorem, if $f \in C^1(D)$, $D \in \mathbb{R}^n$, then

$$f(x) = f(x_0) + D_x f(x_0) \cdot (x - x_0) + R(x)$$

$\forall x_0 \in D$ $\forall x \in B_\varepsilon(x_0)$ for some $\varepsilon > 0$

where

$$\limsup_{\|x - x_0\| \to 0} \frac{\|R(x)\|}{\|x - x_0\|} = 0$$

~~[illegible]~~, Let $\xi \equiv x - x_0$. Then

$A = D_x f(x_0)$ $\qquad \dfrac{d\xi}{dt} = A\xi + R(x_0 + \xi)$

Def

The linear system $\dfrac{dx}{dt} = Ax$ is called the linearization of $\dfrac{dx}{dt} = f(x)$ near the critical point $x = x_* = 0$

Ex

$x(t) \in \mathbb{R}$

$$\begin{cases} \dfrac{dx}{dt} = \lambda x \\ x(0) = x_0 \end{cases} \Rightarrow x(t) = e^{\lambda t} x_0$$

(1) $Re(\lambda) > 0$ - unstable

(2) $Re(\lambda) < 0$ - asymptotically stable

(3) $Re(\lambda) = 0$ - stable for linearization, but inconclusive for a nonlinear system

E.g.

$$\left.\begin{array}{l} \dot{x} = -x^3 \text{ - stable} \\ \dot{x} = x^3 \text{ - unstable} \end{array}\right\}$$

both lead to the same linearization (with $\lambda = 0$) around the critical point $x_* = 0$.

Theorem

The critical point $x_* = 0$ is stable w.r.t linearization $\frac{dx}{dt} = Ax$ iff $Re(\lambda_j) \leq 0$ for each eigenvalue of A and each eigenvalue with $Re(\lambda_j) = 0$ is semi-simple. It is asymptotically stable iff $Re(\lambda_j) < 0$ for each eigenvalue

Proof

$$\begin{cases} \frac{dx}{dt} = Ax \text{ , where } A = D_x f(x_*) \\ x(0) = x_0 \end{cases} \Rightarrow \underline{\phi_t(x_0) = e^{tA} x_0}$$

Transform to the Jordan canonical form $A = PJP^{-1}$ and let $y = P^{-1}x$, then

$$\frac{dy}{dt} = P^{-1}APy = Jy \Rightarrow y(t) = e^{tJ} y_0 = E(t) y_0$$

(A)

We shall use complex-valued forms of P and J.

* For semi-simple eigenvalues $J_j = \lambda_j \Rightarrow E_j(t) = e^{t\lambda_j}$

* for multiple eigenvalues (with multiplicity k_j)

$$J_j = \lambda_j I_{k_j} + N_{k_j} \text{ , where } N_{k_j}^{k_j} = 0$$

and

$$e^{tJ_j} = e^{\lambda_j t}\left[I_{k_j} + tN_{k_j} + \dots + \frac{t^{k_j - 1}}{(k_j - 1)!} N_{k_j}^{k_j - 1}\right]$$

$\forall_{k \geq 1} \ \forall_{\alpha > 0} \ \exists C$ ST $t^k \leq Ce^{\alpha t}$

Therefore if $Re(\lambda_j) < 0$: $\|e^{tJ_j}\| \leq Ce^{(\lambda_j + \alpha)t}$ with $Re(\lambda_j) + \alpha < 0$

(A) $\|x(t)\| \leq \|P\| \|y(t)\| \leq \|P\| \|e^{tJ}\| \|P^{-1}\| \|x_0\| \leq C\|e^{tJ}\|$

need to estimate

(59) $\|e^{tJ}\| \leq C' \max_{k \leq j \leq s} \|e^{tJ_j}\|$

If $\mathrm{Re}(\lambda_j)=0$ and $k_j=1$: $\quad \|e^{tJ_j}\|=1$

$\blacksquare$

Corollary

If all $\mathrm{Re}(\lambda_j)<0$, then there exists $K>0$ and $0>\beta>\max\limits_{1\le j\le n}\mathrm{Re}(\lambda_j)$ ST

$$\|e^{tA}\| \le K e^{\beta t} \quad \forall_{t\ge 0}$$

Ex $\begin{cases} \dot{x}_1 = x_1 - x_1x_2 \\ \dot{x}_2 = -x_1^2 + x_2 \end{cases}$, $f=\begin{bmatrix} x_1(1-x_2) \\ -x_1^2+x_2 \end{bmatrix}$, $D_x f(x)=\begin{bmatrix} 1-x_2 & -x_1 \\ -2x_1 & 1 \end{bmatrix}$

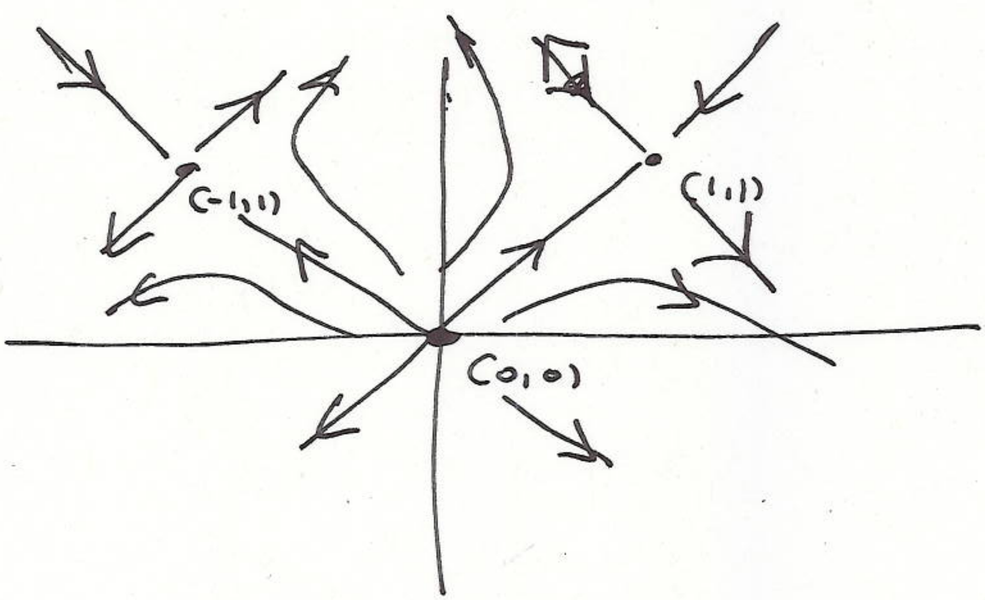
Critical points: $(0,0)$, $(1,1)$, $(-1,1)$

$(0,0)$: $A=\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ – unstable (source)

$(\pm 1,1)$: $A=\begin{bmatrix} 0 & \mp 1 \\ \mp 2 & 1 \end{bmatrix}$, $D(\lambda)=\begin{vmatrix} -\lambda & \mp 1 \\ \mp 2 & 1-\lambda \end{vmatrix} = (\lambda+1)(\lambda-2)$

$\lambda_1=-1$, $\lambda_2=2$

unstable (saddle)



Classification of Critical Points (generalization of the 2D classification)

* all $Re(\lambda_j) < 0$ - sink
* all $Re(\lambda_j) > 0$ - source
* some $Re(\lambda_j) < 0$, some $Re(\lambda_j) > 0$ - saddle
* If all $Re(\lambda_j) \neq 0$ - hyperbolic point
  otherwise, non-hyperbolic
* all $Re(\lambda_j) = 0$ - center point

To be proven:

* sinks asymptotically stable in nonlinear systems
* sources and saddles unstable ——— " ———
* nonhyperbolic points can be either stable or unstable in a nonlinear system.

We shall prove that asymptotic stability of linearized systems implies asymptotic stability in the corresponding nonlinear system in a neighborhood of the critical point.

Given $\begin{cases} \dot{x} = f(x) \\ x(0) = x_0 \end{cases}$

assume: $f(x_*) = 0$ for $x_* \in D \subset \mathbb{R}^n$

$f \in C^1(D)$

$A = D_x f(x_*)$ has all eigenvalues with $Re(\lambda_j) < 0$

WLOG, we'll assume $x_* = 0$