Thm (Lyapunov Linearization Thm)

Under the assumptions above, the critical point $x_1 = 0$ is asymptotically stable and

$\forall_{\varepsilon>0} \; \exists K>0, \beta<0$ ST $\forall_{x_0 \in B_\varepsilon(0)} \; \forall_{t \geq 0}$ $\quad \|x(t)\| \leq K e^{\beta t} \|x_0\|$

So that $\lim_{t\to\infty} \|x(t)\| = 0$

Proof

Since $f \in C^1(D)$, by Taylor's Thm

$$f(x) = Ax + R(x) \quad , \quad \lim_{\|x\|\to 0} \frac{\|R(x)\|}{\|x\|} = 0$$

Then $\dot{x} = Ax + R(x)$

$$x(t) = e^{tA} x_0 + \int_0^t e^{(t-s)A} R(x(s))\,ds$$

$\exists K > 0, \beta_0 > 0, \|e^{tA}\| \leq K e^{-|\beta_0| t} \; \forall_{t \geq 0}$

From the estimates on the remainder $R(x)$

$\forall_{x \in B_\varepsilon(0)}$

$\|R(x)\| \leq k(\varepsilon) \|x\|$

ST $\lim_{\varepsilon\to 0} k(\varepsilon) = 0$

Therefore

$$\|x(t)\| \leq K e^{-|\beta_0| t} \|x_0\| + \int_0^t K e^{-|\beta_0|(t-s)} \|R(x(s))\|\,ds$$

$$e^{|\beta_0| t} \|x(t)\| \leq K \|x_0\| + K \int_0^t e^{|\beta_0| s} \|R(x(s))\|\,ds$$

$$\leq K \|x_0\| + K k(\varepsilon) \int_0^t e^{|\beta_0| s} \|x(s)\|\,ds$$

By Gronwall $\left[r(t) = e^{|\beta_0| t} \|x(t)\| \right.$, $r(t) \leq C_1 + C_2 \int_a^t r(s)\,ds$

we have

$$e^{|\beta_0| t} \|x(t)\| \leq K \|x_0\| e^{K k(\varepsilon) t}$$

$\Rightarrow \|x(t)\| \leq K e^{-(|\beta_0| - Kk(\varepsilon))t} \|x_0\|$

Since $k(\varepsilon) \longrightarrow 0$ as $\varepsilon \to 0$, for any sufficiently small $\varepsilon > 0$ $\exists \beta = -(|\beta_0| - Kk(\varepsilon)) < 0$

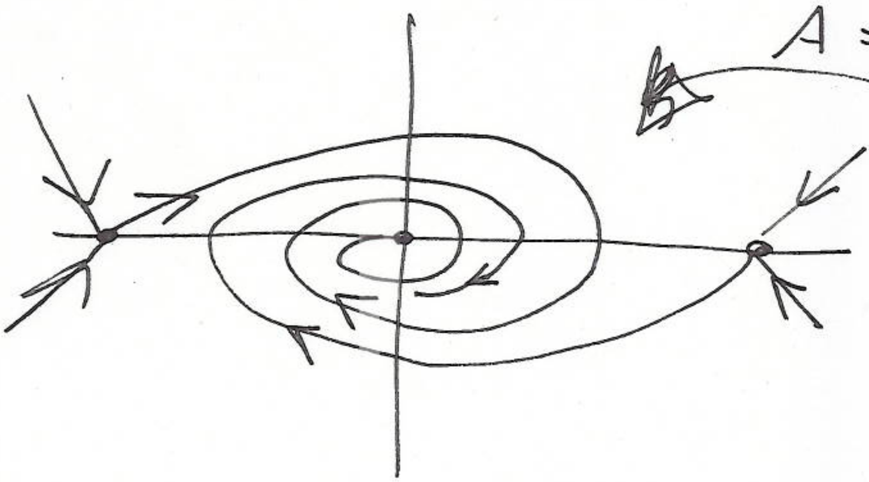
<u>Remark</u>

If there exists at least one eigenvalue with $Re(\lambda_j) > 0$ the linearized system is unstable and the instability persists in $B_\varepsilon(x_0)$ in the nonlinear system.

<u>Ex</u> (damped pendulum)

$$\begin{cases} \ddot{x} + \beta \dot{x} + \sin x = 0 \\ \beta > 0 \end{cases} \Rightarrow \frac{d}{dt}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} x_2 \\ -\beta x_2 - \sin x_1 \end{pmatrix}$$

$$A = \begin{bmatrix} 0 & 1 \\ -1 & -\beta \end{bmatrix}, \quad \lambda_{\pm} = \frac{-\beta \pm i\sqrt{4-\beta^2}}{2}$$



$\beta \in [0,2)$ - complex eigenvalues

$\beta \geq 2$ - real negative eigenvalues

§2.4 LYAPUNOV FUNCTIONS AND NONLINEAR STABILITY – LYAPUNOV'S SECOND METHOD

Lyapunov's second method is particularly suited for systems which in the linearized analysis have:

* $Re(\lambda_j) \leq 0$ for all eigenvalues
* $Re(\lambda_j) = 0$ for ~~some~~ some.

Such ~~a~~ cases are not covered by Lyapunov's first method

<u>Lemma</u>

Let $B_\varepsilon(x_*)$ be a ball of radius ε centered at $x = x_*$ and $B_\varepsilon(x_*) \subset D \subset \mathbb{R}^n$ (local neighborhood of the critical point). Let $V : B_\varepsilon(x_*) \to \mathbb{R}$ be a $C^1(B_\varepsilon(x_*))$ function. If $x = x(t)$ is a solution of $\dot{x} = f(x)$ in $B_\varepsilon(x_*)$ then

$$\frac{dV}{dt} = \nabla V(x) \cdot \frac{dx}{dt} = \nabla V(x) \cdot f(x)$$

<u>Proof</u>

By chain rule

$$\frac{dV}{dt} = \frac{dV}{dx}\frac{dx}{dt} = \nabla V \cdot \frac{dx}{dt} \overset{\text{from the eq.}}{=} \nabla V(x) \cdot f(x)$$

∎

<u>Def</u>

A C^1 function $V : B_\varepsilon(x_*) \to \mathbb{R}$ is called a <u>Lyapunov function</u> for the system $\dot{x} = f(x)$ at the point x_* st $f(x_*) = 0$ if:

① $V(x_*) = 0$

② $V(x) > 0 \quad \forall\, x \in B_\varepsilon(x_*) \setminus \{x_*\}$

③ $\nabla V(x) \cdot f(x) \leq 0 \quad \forall\, x \in B_\varepsilon(x_*)$

<u>Remark</u>

Property ③ implies that $V(x)$ must be <u>nonincreasing</u> along every system trajectory in $B_\varepsilon(x_*)$

<u>Theorem (Lyapunov Stability Theorem)</u>

If there exists a Lyapunov function for the system $\dot{x} = f(x)$ at the critical point x_*, then the critical point is stable.

<u>Ex</u> (so as to motivate the proof)

Harmonic oscillator $\quad \ddot{x} + \omega^2 x = 0$

$$\frac{d}{dt}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} \omega x_2 \\ -\omega x_1 \end{pmatrix} = \begin{pmatrix} 0 & \omega \\ -\omega & 0 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

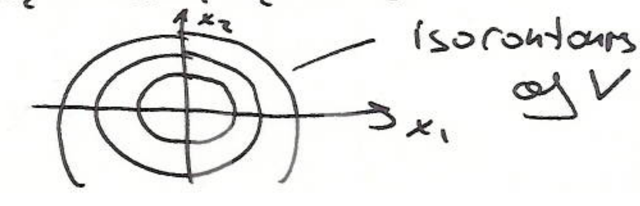
Let $V(x_1, x_2) = \frac{1}{2}(x_1^2 + x_2^2)$

① $V(0,0) = 0$

② $V(x_1, x_2) > 0 \quad \forall\, (x_1, x_2) \in \mathbb{R}^2 \setminus \{(0,0)\}$

③ $\frac{dV}{dt} = x_1 \dot{x}_1 + x_2 \dot{x}_2 = \omega x_1 x_2 - \omega x_1 x_2 = 0$

$\Rightarrow (0,0)$ is stable



Ex $\dot{x} = -x^3$ (from an earlier example ~~re~~ regarding linearized stability)

$V(x) = \frac{1}{4}x^4$

① $V(0) = 0$,

② $V(x) > 0 \quad \forall_{x \in \mathbb{R} \setminus \{0\}}$

③ $\frac{dV}{dt} = x^3 \cdot \dot{x} = -x^6 \leq 0 \quad \forall_{x \in \mathbb{R}}$

$\Rightarrow x = 0$ asymptotically stable