Ex $\dot{x} = -x^3$ (from an earlier example regarding linearized stability)

$V(x) = \frac{1}{4}x^4$

$$\left.\begin{array}{l} ① \ V(0) = 0, \\ ② \ V(x) > 0 \ \forall_{x \in \mathbb{R} \setminus \{0\}} \\ ③ \ \frac{dV}{dt} = x^3 \cdot \dot{x} = -x^6 \le 0 \ \forall_{x \in \mathbb{R}} \end{array}\right\} \Rightarrow x = 0 \text{ asymptotically stable}$$

Proof

Fix $\varepsilon > 0$ and look for $\delta(\varepsilon) > 0$ ST

$$\forall_{x_0 \in B_\delta(x_*)} \quad x(t) = \phi_t(x_0) \in B_\varepsilon(x_*) \ \forall_{t \geq 0}$$

Let $m_\varepsilon = \min\limits_{x \in \partial B_\varepsilon(x_*)} V(x) > 0$ (minimum over the boundary of $B_\varepsilon(x_*)$)

$$\left.\begin{array}{l} V \in C^1(B_\varepsilon(x_*)) \\ V(x_*) = 0 \end{array}\right\} \Rightarrow \exists \, \delta > 0 \text{ ST } V(x) < m_\varepsilon \ \forall_{x \in B_\delta(x_*)}$$

$\dot{V} \le 0 \Rightarrow V(x(t))$ non-increasing function along solution $x(t)$

$$\Rightarrow \forall_{x_0 \in B_\delta(x_*)} \ \forall_{t \geq 0} \quad V(\phi_t(x_0)) \le V(x_0) < m_\varepsilon$$

By contradiction, suppose $\exists \, t_1 \geq 0$ ST $\|\phi_t(x_0)\| = \varepsilon$ (on the boundary). Then $V(\phi_{t_1}(x_0)) \geq m_\varepsilon$. However, this is a contradiction with $V(\phi_t(x_0)) < m_\varepsilon$!
Thus, $\forall_{x_0 \in B_\delta(x_0)}, \forall_{t \geq 0}$ we have $\phi_t(x_0) \in B_\varepsilon(x_*)$ ∎

Remark

If $\frac{dV}{dt} = 0 \quad \forall_{x(t)}$ ST $\frac{dx}{dt} = \underline{f}(x)$, then the trajectory $x(t)$ lies on a level set $V(x) = C$. The function $V(x)$ is called the ~~energy~~ energy of the system $\dot{x} = \underline{f}(x)$.

Ex
$$\begin{cases} \dot{x}_1 = -(x_1^3 + x_2^3) \\ \dot{x}_2 = -(x_1^3 + x_2^3) \end{cases}$$

Let: $V(x_1, x_2) = \frac{1}{4}(x_1^4 + x_2^4) > 0 \quad \forall_{(x_1, x_2) \in \mathbb{R}^2 - \{0\}}$

$V(0,0) = 0$

$$\frac{dV}{dt} = x_1^3 \dot{x}_1 + x_2^3 \dot{x}_2$$
$$= -(x_1^3 + x_2^3)^2 < 0$$
$$\Rightarrow (0,0) - \underline{\text{stable}}$$

Remark

There is no specific rule telling how to construct Lyapunov function $V(x)$, or whether such a function at all exists. This is a major shortcoming of the Lyapunov method. In many problems one can try to work with quadratic forms

$$V(x_1, \dots, x_n) = \sum_{i=1}^{n} \sum_{j=1}^{n} b_{ij} x_i x_j, \text{ where}$$

$[b_{ij}]$ is a $n \times n$ real symmetric matrix.

Theorem

If there exists a continuously differentiable function $V(x)$ such that:

1. $V(x_*) = 0$
2. $V(x) > 0 \quad \forall x \in B_\varepsilon(x_*) \setminus \{x_*\}$
3. $\frac{d}{dt} V(x) < 0, \quad \forall x \in B_\varepsilon(x_*) \setminus \{x_*\}$

then x_* is asymptotically stable.

Proof We will show that:

$\forall_{\varepsilon > 0} \; \exists \, \delta < \varepsilon$ ST if $x_0 \in B_\varepsilon(x_*)$, then

$\phi_{t_0}(x_0) \in B_\delta(x_0)$ for some $t_0(\varepsilon, x_0) >$

Indeed, Let

$$M_\varepsilon = \max_{x \in B_\varepsilon(x_*)} V(x) \quad \text{(over the big ball)}$$

$$m_\delta = \min_{x \in \partial B_\delta(x_*)} V(x) \quad \text{(over the boundary of the small ball)}$$

$$\delta < \varepsilon \implies m_\delta < M_\varepsilon$$

Since $\frac{dV}{dt} \leq -c < 0$ for any $\delta < |x| < \varepsilon$

then $V(\phi_t(x_0)) \leq V(x_0) - ct \leq M_\varepsilon - ct$

Thus, for some $0 < t_0 < t_* = \frac{M_\varepsilon - m_\delta}{c}$

$\phi_{t_0}(x_0) \in B_\delta(x_*)$ and $\lim_{t \to \infty} \phi_t(x_0) = x_*$

(by shrinking $\varepsilon \to 0$)

∎

Ex
$$\begin{cases} \dot{x}_1 = x_2 - x_1(x_1^2+x_2^2) \\ \dot{x}_2 = -x_1 - x_2(x_1^2+x_2^2) \end{cases}, \quad V(x_1,x_2) = \frac{1}{2}(x_1^2+x_2^2)$$
(evidently satisfies properties ① and ②)

$$\frac{d}{dt}V(x_1,x_2) = x_1(x_2 - x_1(x_1^2+x_2^2)) + x_2(-x_1 - x_2(x_1^2+x_2^2))$$
$$= -(x_1^2+x_2^2)^2 < 0 \Rightarrow (0,0) \text{ - asymptotically stable}$$

Indeed, $r = \sqrt{x_1^2+x_2^2}$ reduces the system to $\dot{r} = -r^3 \Rightarrow r=0$ asymptotically stable

Theorem (exponential stability)

If there exist positive constants C_1, C_2, C_3 st
$$C_1\|x\|^2 \leq V(x) \leq C_2\|x\|^2 \text{ and } \frac{d}{dt}V(x) \leq -C_3\|x\|^2$$
then $\|\phi_t(x_0)\| \leq C e^{-\beta t}\|x_0\|$

for some $C>0, \beta>0$, $\forall x_0 \in B_\delta(0)$ for any $\delta>0$ sufficiently small and $\forall t \geq 0$

Proof
$$\frac{dV}{dt} \leq -C_3\|x\|^2 \leq -\frac{C_3}{C_2}V \Rightarrow \text{[by Gronwall (differential form)]}$$
$$\Rightarrow V(\phi_t(x_0)) \leq e^{-\frac{C_3}{C_2}t}V(x_0)$$
therefore
$$\|\phi_t(x_0)\|^2 \leq \frac{1}{C_1}V(\phi_t(x_0)) \leq \frac{1}{C_1}e^{-\frac{C_3}{C_2}t}V(x_0)$$
$$\leq \frac{C_2}{C_1}\|x_0\|^2 e^{-\frac{C_3}{C_2}t}$$
∎

Theorem (Lyapunov Instability Theorem)

Suppose there exists a C^1 function $V(x)$ such that:

① $V(0) = 0$

② $\forall_{\delta > 0} \ \exists x_0 \neq 0, \ \|x_0\| < \delta$ ST $V(x_0) > 0$

③ $\frac{d}{dt} V(x) > 0$ in U, where $U = \{x \in B_r, V(x) > 0\}$ for some $r > 0$ and $B_r \subset D$

Then, the critical point $x = 0$ is unstable.

Proof

The set U is connected due to continuity of V.

$\exists x \in U$ arbitrarily close to $x = 0$ $\Rightarrow$ origin on the border of U

Consider $\dot{x} = f(x)$ with $x(0) = x_0$. Then

$x_0 \in U \Rightarrow V(x(t)) \geq V(x_0) > V(0) \quad \forall_{t \geq 0}$

Let $M \equiv \max_{x \in \partial B_r} V(x)$

Since $\forall_{x \in U} \ \frac{dV}{dt} > 0$ $\exists t_0$ ST $V(\phi_{t_0}(x_0)) > M$

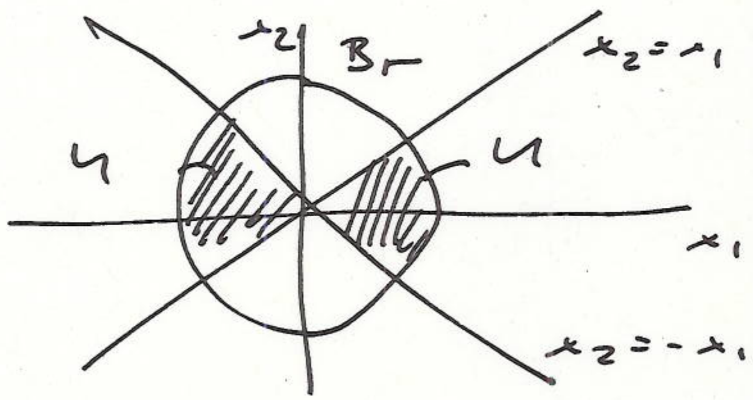
After a finite time $\phi_t(x_0)$ will leave a ball of any radius

Ex

$$\begin{cases} \dot{x}_1 = x_1 \\ \dot{x}_2 = -x_2 \end{cases} \Rightarrow V(x) = \frac{1}{2}(x_1^2 - x_2^2)$$

$$\frac{dV}{dt} = x_1\dot{x}_1 - x_2\dot{x}_2 = x_1^2 + x_2^2 > 0$$

$$U = \{(x_1, x_2) : |x_1| < |x_2|\}$$



$\Rightarrow$ instability

Remark

$V(x)$ need not be sign-definite in the neighbourhood of $x=0$. (less strict requirements than a Lyapunov function in the "stability theorem")

CHAPTER 3 – HYPERBOLIC THEORY

3.1 STABLE & UNSTABLE MANIFOLDS OF DYNAMICAL SYSTEMS

Let $x_* = 0$ be an equilibrium point, i.e. $f(x_*)=0$ and ~~...~~ $A = D_x f(x_*)$ be the Jacobian matrix. If $f \in C^1(B_\delta(x_*))$, then $\frac{dx}{dt} = f(x)$ in $x \in B_\delta(x_*)$

can be represented in the "weakly nonlinear" form

$$\dot{x} = Ax + R(x) \quad \text{where} \lim_{\|x\|\to 0} \frac{\|R(x)\|}{\|x\|} = 0$$

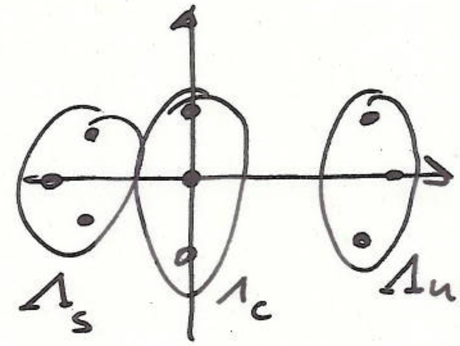
* Stable and Unstable Manifolds in Linearized Systems

Consider matrix A with eigenvalues $\{\lambda_j\}_{j=1}^{n}$ classified into three groups:

$$\Lambda_s = \{(\lambda_1, \dots, \lambda_m),\ \operatorname{Re}\lambda_j < 0\}$$

$$\Lambda_u = \{(\lambda_{m+1}, \dots, \lambda_k),\ \operatorname{Re}\lambda_j > 0\}$$

$$\Lambda_c = \{(\lambda_{k+1}, \dots, \lambda_n),\ \operatorname{Re}\lambda_j = 0\}$$



Consider sets of eigenvectors (including ~~generalized~~ generalized eigenvectors) associated with eigenvalues in each group:

$$E_s = \operatorname{span}\{v_1, \dots, v_m\} \subset \mathbb{R}^n$$

$$E_u = \operatorname{span}\{v_{m+1}, \dots, v_k\} \subset \mathbb{R}^n$$

$$E_c = \operatorname{span}\{v_{k+1}, \dots, v_n\} \subset \mathbb{R}^n$$

Example

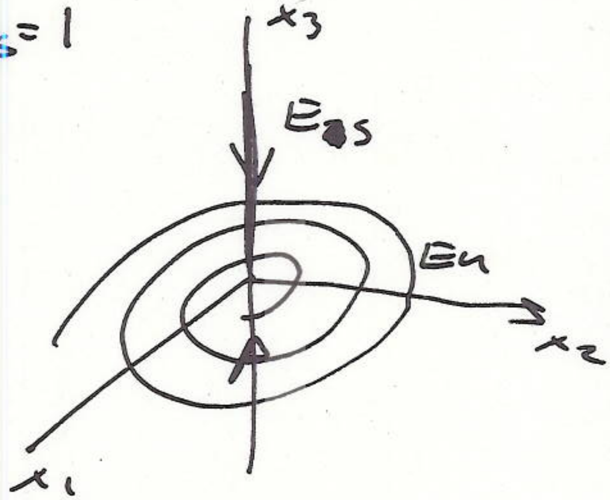
$$\begin{cases} \dot{x}_1 = x_1 + x_2 \\ \dot{x}_2 = -x_1 + x_2 \\ \dot{x}_3 = \qquad\quad -x_3 \end{cases} \Rightarrow A = \begin{bmatrix} 1 & 1 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \Rightarrow \begin{array}{l} \lambda_{1,2} = 1 \pm i \\ v_{1,2} = \begin{bmatrix} 1 \\ \pm i \\ 0 \end{bmatrix} \end{array}$$

$$\lambda_3 = -1,\ v_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$E_s = \{x \in \mathbb{R}^3,\ x_1 = x_2 = 0\}$, $\dim E_s = 1$

$E_u = \{x \in \mathbb{R}^3,\ x_3 = 0\}$, $\dim E_u = 2$

$E_c = \emptyset$



<u>Lemma</u>

Let E_0 be a subspace of $\mathbb{R}^n$ spanned by generalized eigenvectors associated with the eigenvalue λ_0 of A. Then $AE_0 \subset E_0$

<u>Proof</u>

$\forall v \in E_0$, $v = \sum_{j=1}^{k} c_j v_j$. Then $Av = \sum_{j=1}^{k} c_j A v_j$

$$\left.\begin{array}{ll} v_j \text{ - eigenvector} & A v_j = \lambda_j v_j \\ v_j \text{ - generalized eigenvector} & A v_j = \lambda_0 v_j + v_{j-1} \end{array}\right\} \Rightarrow A v_j \in E_0 \Rightarrow AE_0 \subset E_0$$

∎

<u>Def</u>

A subspace $S \subset \mathbb{R}^n$ is <u>invariant</u> w.r.t the phase flow $\phi_t(x)$ if $\forall_{x \in S}$ $\phi_t(x) \subset S$ $\forall_{t \in I \subset \mathbb{R}}$

<u>Ex</u>

A set of critical points of $\dot{x} = f(x)$ is invariant w.r.t $\phi_t(x)$.

<u>Ex</u>

The level sets of the energy function are an invariant set of a <u>conservative</u> dynamical system

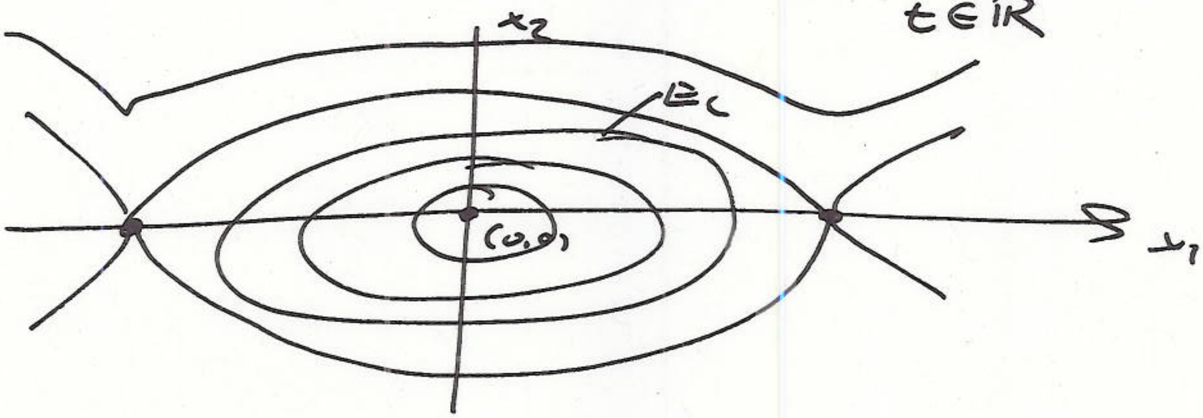
$$\ddot{x} + \sin x = 0 \iff \begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = -\sin x_1 \end{cases}$$

$$E(x_1, x_2) = \frac{1}{2} x_2^2 - \cos x_1 = C$$

$$\frac{dE}{dt} = x_2 \dot{x}_2 + \sin x_1 \dot{x}_1 = \sin x_1 (-x_2 + \dot{x}_1) = 0$$

if $x \in S_C = \left\{ x \in \mathbb{R}^2, \frac{1}{2} x_2^2 - \cos x_1 = C \right\}$ for $t = 0$

then $x \in S_C \ \forall_{t \in \mathbb{R}}$



Thm

The subspaces E_s, E_u, E_c are invariant w.r.t $\phi_t(x) = e^{tA} x \ \forall_{t \in \mathbb{R}}$ and the $\overbrace{\text{decomposition}}^{}$ is unique.

$$(\mathbb{R}^n = E_s \oplus E_u \oplus E_c)$$

Proof

The generalized eigenvectors of A form a basis for $\mathbb{R}^n$. Furthermore

$$\mathbb{R}^n = E_s \oplus E_u \oplus E_c \iff$$

$$\forall_{x \in \mathbb{R}^n} \ x = \underbrace{\sum_{j=1}^{m} c_j v_j}_{\in E_s} + \underbrace{\sum_{j=m+1}^{k} c_j v_j}_{\in E_u} + \underbrace{\sum_{j=k+1}^{n} c_j v_j}_{\in E_c} \text{ - unique decomposition}$$

Assume $x_0 \in E_s$, then $x_0 = \sum_{j=1}^{m} c_j v_j$
and

$$x(t) = \phi_t(x_0) = e^{tA}x_0 = \lim_{n\to\infty}\left[I + tA + \dots + \frac{t^n}{n!}A\right]x_0$$

$$= \sum_{j=1}^{m} c_j \lim_{n\to\infty}\left[I + tA + \dots + \frac{t^n}{n!}A^n\right]v_j$$

Since $A^k v_j \in E_s$ $\forall_{k \geq 1}$, the sum converges absolutely and ~~tat~~ uniformly. Moreover, since E_s is complete, then $e^{tA}x_0 \in E_s$. Analogously for E_u and E_c.

$\blacksquare$

<u>Ex</u>

$$\begin{cases} \dot{x}_1 = 2x_1 + x_2 \\ \dot{x}_2 = 2x_2 \\ \dot{x}_3 = x_3 \end{cases} \Rightarrow A = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

$\lambda_1 = \lambda_2 = 2$, $\lambda_3 = -1$ Thus:

$v_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$, $v_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ $v_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

$E_u = \{x \in \mathbb{R}^3, x_3 = 0\}$

$E_s = \{x \in \mathbb{R}^3, x_1 = x_2 = 0\}$

E.g., if $x_3 = 0$ at $t = 0 \Rightarrow x_3 = 0 \ \forall_{t \geq 0}$

<u>Thm</u>

Let $x_0 \in E_s$. Then, $\exists\, C > 0$ and $\beta > 0$ ST $e^{tA}x_0 \in E_s$, $\|e^{tA}x_0\| \leq C e^{-\beta t}\|x_0\|$, $\forall t \geq 0$ and

$$\lim_{t\to\infty} \phi_t(x) = 0$$

Proof

In canonical variables $(y = P^{-1}x)$, $PAP^{-1} = J$ has a $k \times k$ block with all eigenvalues $Re(\lambda_j) < 0$. Since E_s is an invariant manifold, the result follows from the previous thm on stable matrices.

Thm

Let $x_0 \in E_u$, $\exists\, C > 0,\ \beta > 0$ st $e^{tA}x_0 \in E_u$, $\| e^{tA}x_0 \| \leq C e^{\beta t} \|x_0\|$, $\forall_{t \leq 0}$ and $\lim_{t \to -\infty} \phi_t(x) = 0$

Proof as before

Because of these properties, the sets E_s and E_u are referred to as the stable and unstable manifolds of the linear system.