Proof

In canonical variables $(y = P^{-1}x)$, $PAP^{-1} = J$ has a $k \times k$ block with all eigenvalues $Re(\lambda_j) < 0$. Since E_s is an invariant manifold, the result follows from the previous thm on stable matrices.

Thm

Let $x_0 \in E_u$, $\exists C > 0$, $\beta > 0$ st $e^{tA}x_0 \in E_u$, $\|e^{tA}x_0\| \leq Ce^{\beta t}\|x_0\|$, $\forall_{t \leq 0}$ and $\lim_{t \to -\infty} \phi_t(x) = 0$

Proof as before

Because of these properties, the sets E_s and E_u are referred to as the stable and unstable manifolds of the linear system.

* **Invariant Manifolds for Nonlinear Dynamical Systems in a Neighbourhood of a Critical Point**

Def

Let x_* be a critical point of $f(x)$. $W_s(x_*)$ is called a stable manifold for x_* if

$\forall_{x_0 \in W_s(x_*)}$ $\phi_t(x_0) \in W_s(x_*)$ $\forall_{t \geq 0}$ and $\lim_{t \to \infty} \phi_t(x_0) = x_*$

$W_u(x_*)$ is called an unstable manifold for x_* if

$\forall_{x_0 \in W_u(x_*)}$ $\phi_t(x_0) \in W_u(x_*)$ $\forall_{t \geq 0}$ and $\lim_{t \to -\infty} \phi_t(x_0) = x_*$

Ex
$$\begin{cases} \dot{x}_1 = x_1 \\ \dot{x}_2 = -x_2 + x_1^2 \end{cases} \qquad \begin{aligned} E_s &= \{ x_2 \in \mathbb{R},\ x_1 = 0 \} \\ E_u &= \{ x_1 \in \mathbb{R},\ x_2 = 0 \} \end{aligned}$$

$$\left. \begin{aligned} &x_1 = 0 \text{ is invariant} \\ &\text{w.r.t the evolution } \dot{x}_2 = x_2 \end{aligned} \right\} \Rightarrow W_s(0) = E_s$$

$x_2 = 0$ is not invariant

$\dot{x}_1 = x_1$ gives the dynamics on $W_u(0)$.

Let $W_u(0) = \{ (x_1, x_2) \in \mathbb{R}^2,\ x_2 = \psi(x_1) \}$

Need to find the function $\psi : \mathbb{R} \to \mathbb{R}$

Using the second equation

$$\psi'(x_1)\dot{x}_1 = -\psi(x_1) + x_1^2 \Rightarrow x_1 \frac{d\psi}{dx_1} + \psi(x_1)$$
$$= \frac{d}{dx_1}\left[x_1 \psi(x_1) \right] = x_1^2$$

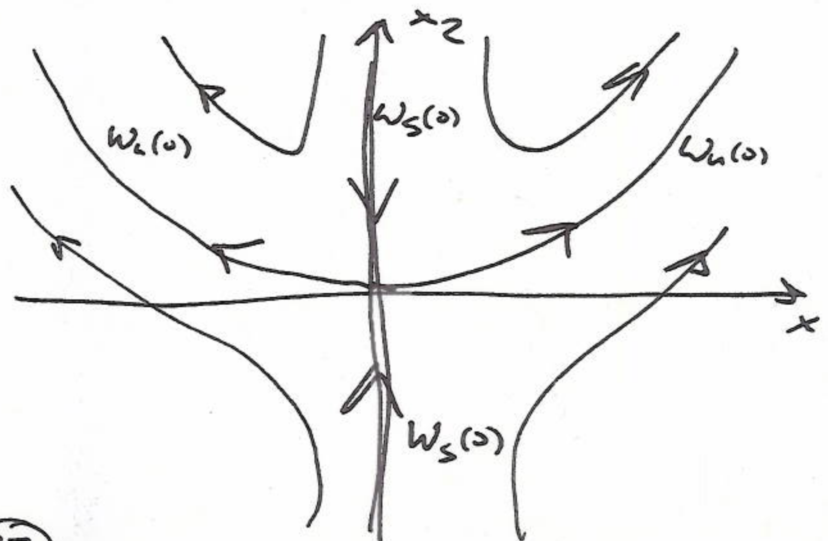
Integrating

$$x_1 \psi(x_1) = \frac{1}{3} x_1^3 + C \Rightarrow \psi(x_1) = \frac{1}{3} x_1^2 + \frac{C}{x_1},$$

Fix the constant C : $\psi(0) = 0 \Rightarrow C = 0$

Thus

$$W_u(0) = \left\{ (x_1, x_2) \in \mathbb{R}^2 : x_2 = \frac{1}{3} x_1^2 \right\}$$



We note that $W_s(0)$ and $W_u(0)$ are <u>invariant sets</u> for all $t \in \mathbb{R}$. They are tangent to E_s and E_u near the origin and have the same dimensions.

<u>Assumptions of the Main Theorem</u>

① $x_* = 0$ is the critical point

② x_* is a hyperbolic point ($\Lambda_c = \{\}$)

③ $A = D_x f(0) = \begin{bmatrix} P & 0 \\ 0 & Q \end{bmatrix}$, where

any matrix A can be transformed to this form via $C^{-1} A C$

* P is an $m \times m$ block with $\mathrm{Re}(\lambda_j) < 0$ (corresponding to E_s)

* Q is an $(n-m) \times (n-m)$ block with $\mathrm{Re}(\lambda_j) > 0$ (corresponding to E_u)

* $\dot{x} = f(x) = Ax + F(x)$, where $F \in C^1(B_\varepsilon, \mathbb{R}^n)$

and $\lim_{\|x\| \to 0} \frac{\|F(x)\|}{\|x\|} = 0$ $\quad (F(x) = f(x) - Ax)$

Note that this implies that $F(0) = 0$, $D_x F(0) = 0$

We also have

$$\forall_{\varepsilon > 0} \; \exists K(\varepsilon) > 0 \text{ ST } \|F(x)\| \leq K(\varepsilon) \|x\| \; \forall_{x \in B_\varepsilon(0)}$$

and

$$\lim_{\varepsilon \to 0} K(\varepsilon) = 0$$

Theorem (Stable Manifold)

Under the above assumptions, there exists a m-dimensional invariant manifold $W_s(0)$ in $B_\varepsilon(0)$ for some small $\varepsilon>0$ which is tangent to E_s and

$$\forall_{x \in W_s(0)} \quad \phi_t(x_0) \in W_s(0) \quad \forall_{t \geq 0} \quad \text{and}$$

$$\lim_{t\to\infty} \phi_t(x_0) = 0$$

Proof

Let $x = \binom{u}{v}$, where $u \in \mathbb{R}^{m}$, $v \in \mathbb{R}^{(n-m)}$

$$F(x) = \begin{pmatrix} g(u,v) \\ h(u,v) \end{pmatrix}$$

such that
$$\begin{cases} \dot{u} = Pu + g(u,v) & , u(0) = u_0 \\ \dot{v} = Qv + h(u,v) & , v(0) = v_0 \end{cases}$$

Thus, we have
$$\begin{cases} u(t) = e^{tP} u_0 + \int_0^t e^{(t-s)P} g(u(s),v(s))\,ds \\ v(t) = e^{tQ} v_0 + \int_0^t e^{(t-s)Q} h(u(s),v(s))\,ds \end{cases}$$

We shall ~~[illegible]~~ demonstrate that the solutions remain bounded as $t \to \infty$. In particular, this implies that $\lim_{t\to\infty} e^{t\alpha} v(t) = 0$

Therefore, from the second equation we have

$$v_0 \cancel{+} \int_0^\infty e^{-sQ} h(u(s),v(s))\,ds = 0$$

$$v_0 = \int_\infty^0 e^{-sQ} h(u(s),v(s))\,ds$$

and $$v(t) = \int_\infty^t e^{(t-s)Q} h(u(s),v(s))\,ds \quad \text{for } t \geq 0$$

Defining $U(t) = \begin{bmatrix} e^{tP} & 0 \\ 0 & 0 \end{bmatrix}$, $V(t) = \begin{bmatrix} 0 & 0 \\ 0 & e^{tQ} \end{bmatrix}$

we can write

$$x(t) = U(t)x_0 + \int_0^t U(t-s)F(x(s))\,ds + \int_{-\infty}^t V(t-s)F(x(s))\,ds$$

We shall prove that Picard's iterations

$$x^{(0)}(t) = U(t)x_0$$

$$x^{(k+1)}(t) = U(t)x_0 + \int_0^t U(t-s)F(x^{(k)}(s))\,ds + \int_{+\infty}^t V(t-s)F(x^{(k)}(s))\,ds \qquad k \geq 0$$

satisfy the following properties:

1. $x^{(k)}(t)$ exists in $B_\varepsilon(0)$ for $t \in \mathbb{R}^+$ if $x_0 \in B_\delta(0)$ for any $\varepsilon > 0$ and some $\delta(\varepsilon) > 0$
2. $\{x^{(k)}(t)\}_{k \in \mathbb{N}}$ is a Cauchy sequence in $C(\mathbb{R}^+, B_\varepsilon(0) \subset \mathbb{R}^n)$

③ The limiting function $\tilde{x}_*(t) = \lim_{k\to\infty} x^{(k)}(t) \in B_\varepsilon(0)$

is the unique solution of the given integral equation and

$$\tilde{x}_*(t) \in B_\varepsilon(0), \quad \forall\, t \geq 0$$

$$\lim_{t\to\infty} \tilde{x}(t) = 0$$

We shall use the estimates:

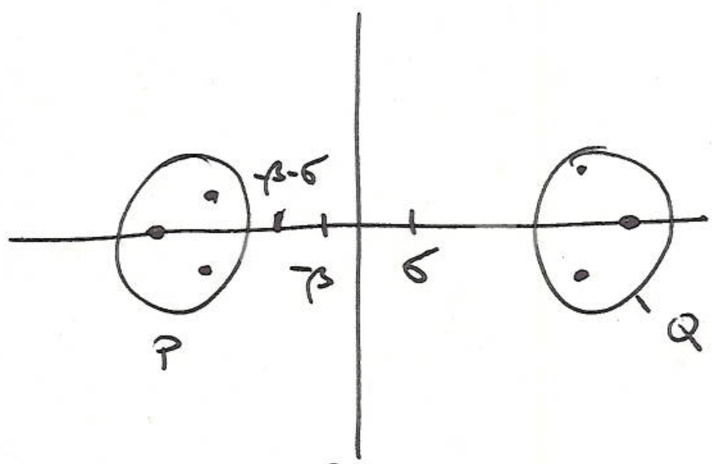
$\exists\, M>0,\ \beta>0,\ \sigma>0$ ST

$$0 < \sigma < \min_{k+1\leq j\leq n} (Re(\lambda_j)) \qquad \text{(eigenvalues of the Q block)}$$

$$\max_{1\leq j\leq k} (Re(\lambda_j)) < -(\beta+\sigma) < 0 \qquad \text{(eigenvalues of the P block)}$$

$\forall\, t\geq 0$ $\quad \|U(t)\| \leq M e^{-(\beta+\sigma)t}$

$\quad \|V(t)\| \leq M e^{\sigma t}$



① will use mathematical induction

$$\|x^{(0)}(t)\| \leq \|U(t)\| \|x_0\|$$

$$\leq M e^{-(\beta+\sigma)t} \|x_0\| \leq 2M e^{-\beta t} \|x_0\|$$

So that if $x_0 \in B_\delta(0) \Rightarrow x^{(0)}(t) \in B_\varepsilon(0)$ with $\varepsilon = 2M\delta$

By induction, we will assume that

$$\|x^{(k)}(t)\| \leq 2M e^{-\beta t} \|x_0\| \quad \forall\, t \geq 0$$

Then

$$\|x^{(k+1)}(t)\| \le \|U(t)\|\,\|x_0\| + \int_0^t \|U(t-s)\|\cdot\|F(x^{(k)}(s))\|\,ds$$

$$+ \int_t^\infty \|V(t-s)\|\,\|F(x^{(k)}(s))\|\,ds$$

$$\le M e^{-\beta t}\|x_0\| + MK(\varepsilon)\int_0^t e^{-(\beta+\sigma)(t-s)}\|x^{(k)}(s)\|\,ds$$

$$+ MK(\varepsilon)\int_t^\infty e^{\sigma(t-s)}\|x^{(k)}(s)\|\,ds = \left|\begin{array}{l}\text{use} \\ \text{induction} \\ \text{hypothesis}\end{array}\right|$$

$$= M e^{-\beta t}\|x_0\| + 2M^2K(\varepsilon)\, e^{-(\beta+\sigma)t}\|x_0\| \int_0^t e^{\sigma s}\,ds$$

$$+ 2M^2K(\varepsilon)\, e^{\sigma t}\|x_0\| \int_t^\infty e^{-(\beta+\sigma)s}\,ds$$

$$\le M e^{-\beta t}\|x_0\| \underbrace{\left(1 + \frac{2MK(\varepsilon)}{\sigma} + \frac{2MK(\varepsilon)}{(\beta+\sigma)}\right)}$$

Since $K(\varepsilon)$ is small for ε small enough

$$1 + \frac{2MK(\varepsilon)}{\sigma} + \frac{2MK(\varepsilon)}{(\beta+\sigma)} < 2$$

$$< 2M e^{-\beta t}\|x_0\| \Rightarrow \text{induction hypothesis verified}$$

and

$$\forall_{x_0 \in B_\delta(0)} \quad x^{(k)}(t) \in B_\varepsilon(0) \quad \forall_{t \ge 0}, \ \varepsilon = 2M\delta$$

② Check that $\|x^{(1)}(t) - x^{(0)}(t)\| \le \frac{1}{2} M e^{-\beta t}\|x_0\|$

and assume that

(induction hypothesis) $\|x^{(k)}(t) - x^{(k-1)}(t)\| \le \frac{1}{2^k} M e^{-\beta t}\|x_0\|$

Then,

$$\|x^{(k+1)}(t) - x^{(k)}(t)\| \le \int_0^t \|U(t-s)\|\cdot\|F(x^{(k)}(s)) - F(x^{(k-1)}(s))\|ds$$

$$+ \int_t^\infty \|V(t-s)\|\,\|F(x^{(k)}(s)) - F(x^{(k-1)}(s))\|ds$$

$$\le M e^{-(\beta+\sigma)t} K(\varepsilon) \int_0^t e^{s(\beta+\sigma)} \frac{1}{2^k} M e^{-\beta s}\|x_0\| ds$$

$$+ M e^{\sigma t} K(\varepsilon) \int_t^\infty e^{-\sigma s} \frac{1}{2^k} M e^{-\beta s}\|x_0\| ds$$

$$\le \frac{1}{2^k} M^2 K(\varepsilon) e^{\beta t}\|x_0\| \left(\frac{1}{\sigma} + \frac{1}{\beta+\sigma}\right)$$

$$\le \frac{1}{2^{k+1}} M \|x_0\| e^{\beta t}$$

for sufficiently small ε and $K(\varepsilon)$ and $\forall\, t \ge 0$

Therefore $\{x^{(k)}(t)\}_{k\in\mathbb{N}}$ is a Cauchy sequence in $C(\mathbb{R}^+, B_\varepsilon(0) \subset \mathbb{R}^n)$ converging to some $\tilde{x}(t)$ as $k \to \infty$ uniformly on $t \in \mathbb{R}^+$

③ Due to uniform convergence, $\tilde{x}(t)$ also solves the integral equation

Also

$$\lim_{t\to\infty} \|\tilde{x}(t)\| = \lim_{t\to\infty}\lim_{k\to\infty} \|x^{(k)}(t)\|$$

$$= \left| \begin{array}{l} \text{owing to uniform} \\ \text{on } t\in\mathbb{R}^+ \\ \text{convergence} \end{array} \right| = \lim_{k\to\infty}\lim_{t\to\infty} \|x^{(k)}(t)\| = 0$$

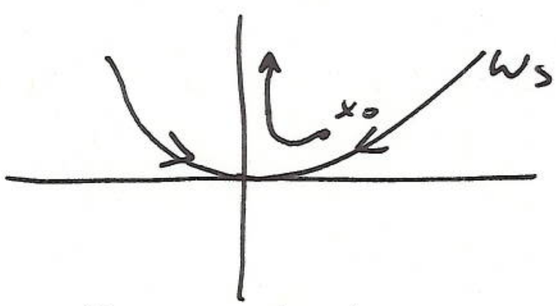
since $\|x^{(k)}(t)\| \leq 2Me^{-\beta t}\|x_0\|$

with $\beta > 0$

Remarks

* $\exists\, x_0 \in B_\varepsilon(0)$, but $x_0 \notin W_s(0)$

then $\phi_t(x_0) \notin B_\varepsilon(0)$ for some $t \geq t_0 > 0$



Trajectory leaves $B_\varepsilon(0)$ in finite time

* The stable manifold $W_s(0)$ can be computed using the following parametrization

$$W_s(0) = \left\{ (x_1, \dots, x_m) \in \mathbb{R}^m,\ x_j = \psi_j(x_1, \dots, x_m),\ j = m+1, \dots, n \right\}$$

where $\psi_j \in C^1(B_\varepsilon(0), \mathbb{R}^{n-m})$

ST

$\psi(0) = 0$ - follows from uniqueness of $v_0 = 0$ if $u_0 = 0$

$D_x \Psi(0)$ – because $W_s(0)$ is tangent to E_s, ((g,h) are superlinear functions of (u,v))

The parameters of $W_s(0)$ are the initial values of $u_0 \in \mathbb{R}^m$, whereas $v_0 = \Psi(u_0)$ is uniquely defined

* If F is $C^k(B_\varepsilon(0), \mathbb{R}^{n-m})$ then
  $\Psi \in C^k(B'_\varepsilon(0), \mathbb{R}^{n-m})$

  If F is real-analytic, then Ψ is real-analytic too.

* All results for $W_u(0)$ can be obtained by reversing $t \to -t$
* One can also include the center manifold $W_c(0)$ (the center manifold thm)

<u>Ex</u> (from the textbook, but solved using a different method)

$$\begin{cases} \dot{x}_1 = -x_1 - x_2^2 \\ \dot{x}_2 = x_2 + x_1^2 \end{cases} \Rightarrow A = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \Rightarrow \text{origin is an (unstable) saddle}$$

Stable subspace

$E_s = \{ x_1 \in \mathbb{R} \ , \ x_2 = 0 \}$

Local stable manifold near origin

$W_s = \{ x_1 \in \mathbb{R}, \ x_2 = \Psi(x_1) \}$

Dynamics on $W_s(0)$

$$\dot{x}_1 = -x_1 - \psi^2(x_1)$$

$$\psi'(x_1)\,\dot{x}_1 = \psi(x_1) + x_1^2$$

$$\psi'(x_1)\left(-x_1 - \psi^2(x_1)\right) = \psi(x_1) + x_1^2$$ (nonlinear equation for $\psi(x_1)$)