MATH 741 – FINAL EXAM

SOLUTIONS

(2) Duffing equation $\ddot{x} + x - x^3 = 0$

$$\left.\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -x_1 + x_1^3 = -q(x_1) \end{aligned}\right\} \Rightarrow$$

linearization at the origin

$$A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \Rightarrow \text{eigenvalues } \lambda_{1/2} = \pm i$$

The origin is not a hyperbolic point and we need to use Lyapunov's second method.

Note that $q(x)$ satisfies conditions of Example 4 on page 130 in Perko, i.e.

$$q(x) = x(1-x^2) \qquad x\,q(x) > 0 \quad \forall\, \substack{x \neq 0 \\ |x| < 1}$$

Therefore $V(x_1,x_2) = \dfrac{x_2^2}{2} + \displaystyle\int_0^{x_1} q(x)\,dx$

$$= \frac{x_2^2}{2} + \frac{x_1^2}{2} - \frac{x_1^4}{4}$$

$$= \frac{1}{2}\left(x_2^2 + x_1^2 - \frac{x_1^4}{2}\right)$$

(P1) $V(0,0) = 0$

(P2) Note that

$$V(x_1,x_2) \geq \frac{1}{2}\left(x_1^2 - \frac{x_1^4}{2}\right) = \frac{x_1^2}{4}(2 - x_1^2) > 0 \quad \forall\, |x| < \sqrt{2}$$

(P3)

$$\frac{dV}{dt} = x_2 \dot{x}_2 + q(x_1)\dot{x}_1 = x_2(-q(x_1)) + q(x_1)x_2$$
$$= 0$$

Thus, the origin is stable, but not necessarily asymptotically stable.

(3) $\begin{cases} \dot{x} = f(x) \\ x(0) = x_0 \end{cases}$ Define $\phi(t) = \left.\frac{\partial x(t,y)}{\partial y}\right|_{y=x_0}$

which satisfies the Cauchy problem

$\phi(t) \in \mathbb{R}^{n \times n}$

$$\begin{cases} \dot{\phi}(t) = A(t)\,\phi(t) \\ \phi(0) = I \end{cases}$$

where $A(t) = D_x f(x(t;x_0))$

Use Liouville's formula

$$\det \phi(t) = \exp\left[\int_0^t \operatorname{tr} A(s)\,ds\right] \det \phi(0)$$

Note that

$\det \phi(0) = \det I = 1$

$A_{ij} = \frac{\partial f_i}{\partial x_j}(x(t;x_0))$

therefore

$$\operatorname{Tr} A = \sum_{j=1}^{n} A_{jj} = \sum_{j=1}^{n} \frac{\partial f_i}{\partial x_i} = \nabla \cdot f(x(t,x_0))$$

(2)

and

$$\det\left[\frac{\partial x(t;y)}{\partial y}\right]\bigg|_{y=x_0} = \exp\left[\int_0^t \nabla\cdot f(x(s;x_0))\,ds\right]$$

∎

(5) $\begin{cases} \dot{x}_1 = x_1 + 6x_2 + x_1 x_2 \\ \dot{x}_2 = 4x_1 + 3x_2 - x_1^2 \end{cases}$

Linearization (ξ_1, ξ_2) $\begin{cases} \dot{\xi}_1 = \xi_1 + 6\xi_2 \\ \dot{\xi}_2 = 4\xi_1 + 3\xi_2 \end{cases}$,

$$A = \begin{bmatrix} 1 & 6 \\ 4 & 3 \end{bmatrix} \Rightarrow \begin{matrix} \lambda_1 = -3 & , v_1 = [3 \ -2]^T \\ \lambda_2 = 7 & , v_2 = [1 \ 1]^T \end{matrix}$$

Therefore

$$P = [v_1 \ v_2] = \begin{bmatrix} 3 & 1 \\ -2 & 1 \end{bmatrix} , \quad P^{-1} = \frac{1}{5}\begin{bmatrix} 1 & -1 \\ 2 & 3 \end{bmatrix}$$

and $P^{-1}AP = B = \begin{bmatrix} -3 & 0 \\ 0 & 7 \end{bmatrix}$

$$x = Py \Rightarrow \begin{cases} x_1 = 3y_1 + y_2 \\ x_2 = -2y_1 + y_2 \end{cases}$$

$$\frac{dy}{dt} = By + P^{-1}N(Py), \text{ where } N(x) = \begin{bmatrix} x_1 x_2 \\ -x_1^2 \end{bmatrix}$$

$$P^{-1}N(Py) = \frac{1}{5}\begin{bmatrix} 1 & -1 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} (3y_1+y_2)(-2y_1+y_2) \\ -(3y_1+y_2)^2 \end{bmatrix}$$

$$= \frac{1}{5}\begin{bmatrix} 3y_1^2 + 7y_1y_2 + 2y_2^2 \\ -39y_1^2 - 16y_1y_2 - y_2^2 \end{bmatrix}$$

Therefore

$$\dot{y}_1 = -3y_1 + \frac{1}{5}(3y_1^2 + 7y_1y_2 + 2y_2^2)$$

$$\dot{y}_2 = 7y_2 - \frac{1}{5}(39y_1^2 + 16y_1y_2 + y_2^2)$$