FINAL EXAMINATION

Due: 9:00am on Friday, December 18

Instructions:

- The examination consists of *five* questions, each worth the number of points indicated, for a total value of 24 points.
- All answers should be supported with suitable justifications and/or proofs.
- Submission: electronically (via Email to the instructor) no later than the time indicated above
  - Problems #1 and #4: please submit your solutions as a Maple worksheet; make sure that answers to the different tasks in a given problem are clearly identified,
  - Problems #2, #3 and #5: please submit your solutions as a PDF file (no MS Word or PowerPoint files, please!).
- Late submissions and submissions which do not comply with these guidelines will not be accepted.

---

1. You are given the matrix exponential

$$e^A = \begin{bmatrix} \frac{1}{2}(e^2+1) & 0 & \frac{1}{2}(e^2-1) \\ 0 & e & 0 \\ \frac{1}{2}(e^2-1) & 0 & \frac{1}{2}(e^2+1) \end{bmatrix}.$$

What is the matrix A?
[3 points]

2. Determine the stability of the origin in the Duffing equation

$$\ddot{x} + x - x^3 = 0.$$

[4 points]

3. The Cauchy problem

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}),$$
$$\mathbf{x}(0) = \mathbf{x}_0,$$

where $\mathbf{x}(t), \mathbf{x}_0 \in \mathbb{R}^n$ and $\mathbf{f} : \mathbb{R}^n \to \mathbb{R}^n$, has a unique solution $\mathbf{x} = \mathbf{x}(t; \mathbf{x}_0)$ for all $t \in [0, a)$ for some $a > 0$. Prove the following identity

$$\det \left[\frac{\partial \mathbf{x}(t; \mathbf{y})}{\partial \mathbf{y}} \right] \Bigg|_{\mathbf{y}=\mathbf{x}_0} = \exp \left[\int_0^t \boldsymbol{\nabla} \cdot \mathbf{f}(\mathbf{x}(s; \mathbf{x}_0)) \, ds \right] \quad \forall t \in [0, a),$$

where $\left[\frac{\partial \mathbf{x}(t;\mathbf{y})}{\partial \mathbf{y}} \right]_{i,j} = \frac{\partial x_i(t;\mathbf{y})}{\partial y_j}$, $i, j = 1, \ldots, n$.
[4 points]

4. Consider the following system of nonlinear ODEs

$$\begin{cases} \dot{x}_1 = -x_1 - x_1 x_2 \\ \dot{x}_2 = x_2 + x_1^2 \end{cases}. \tag{1}$$

Using Maple construct Picard's iterations to compute approximations of

(a) the stable manifold $W_s(0,0)$, and

(b) the unstable manifold $W_u(0,0)$

in the neighborhood the origin $(x_1, x_2) = (0,0)$ (see reference [1, Example 2 in Section 2.7] and lecture notes for details). The manifolds should be approximated at least with the accuracy of $\mathcal{O}(x_1^6)$ and $\mathcal{O}(x_2^6)$, respectively. Plot these approximations of the stable and unstable manifolds in the phase plane (x_1, x_2) assuming that $x_1, x_2 \in [-1, 1]$. For comparison, generate another plot with the vector field $\mathbf{f}(x_1, x_2)$ of system (1) and a number of trajectories chosen to lie close to the manifolds $W_s(0,0)$ and $W_u(0,0)$ (in this plot also use $x_1, x_2 \in [-1, 1]$). The second plot can be created using Maple commands discussed in reference [2, Chapter 2]. Then

(c) using the approximations computed above construct the reduction of system (1) to the stable manifold $W_s(0,0)$, i.e., make the substitution $x_2 = \Psi(x_1)$ in the first equation, and obtain numerically the corresponding solution $x_1 = x_1(t)$ of the reduced system with initial condition $x_1(0) = -1$; plot this solution for $t \in [0, 1]$ (the last step can be performed using the command `DEplot`).

[4 + 3 + 2 = 9 points]

5. Write the system

$$\begin{cases} \dot{x}_1 = x_1 + 6x_2 + x_1 x_2 \\ \dot{x}_2 = 4x_1 + 3x_2 - x_1^2 \end{cases}$$

in the form

$$\dot{\mathbf{y}} = B\,\mathbf{y} + \mathbf{G}(\mathbf{y})$$

where

$$B = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix},$$

$\lambda_1 < 0$, $\lambda_2 > 0$, $\mathbf{G}(\mathbf{y})$ is quadratic in y_1 and y_2 and linearization is performed around the origin.
[4 points]

References

[1] L. Perko, *Differential Equations and Dynamical Systems*, Third Edition, Springer, (2000), ISBN 0–387–95116–4

[2] S. Lynch, *Dynamical Systems with Applications Using MAPLE*, Second Edition, Birkhäuser, (2010). ISBN 978–0–8176–4389–8