

Agenda

Chebyshev Approximation (I)

- Galerkin Approach

- Collocation Approach

- Reciprocal Relations & Economization of Power Series

Chebyshev Approximation (II)

- Spectral Differentiation

- Differentiation in Real Space

Implementation of Boundary Conditions

- Galerkin Approach & Basis Recombination

- Galerkin Approach & Tau Method

- Collocation Method

- ▶ Consider an approximation of $u \in L^2_\omega(I)$ in terms of a **TRUNCATED CHEBYSHEV SERIES** $u_N(x) = \sum_{k=0}^N \hat{u}_k T_k(x)$
- ▶ Cancel the projections of the residual $R_N = u - u_N$ on the $N + 1$ first basis function (i.e., the Chebyshev polynomials)

$$(R_N, T_l)_\omega = \int_{-1}^1 \left(u T_l \omega - \sum_{k=0}^N \hat{u}_k T_k T_l \omega \right) dx = 0, \quad l = 0, \dots, N$$

- ▶ Taking into account the orthogonality condition, expressions for the Chebyshev expansions coefficients are obtained

$$\hat{u}_k = \frac{2}{\pi c_k} \int_{-1}^1 u T_k \omega dx,$$

which can be evaluated using, e.g., the

GAUSS-LOBATTO-CHEBYSHEV QUADRATURES .

- ▶ **QUESTION** — What happens on the boundary?

Theorem

Let $P_N : L^2_\omega(I) \rightarrow \mathbb{P}_N$ be the orthogonal projection on the subspace \mathbb{P}_N of polynomials of degree $\leq N$. For all μ and σ such that $0 \leq \mu \leq \sigma$, there exists a constant C such that

$$\|u - P_N u\|_{\mu, \omega} < CN^{e(\mu, \sigma)} \|u\|_{\sigma, \omega}$$

$$\text{where } e(\mu, \sigma) = \begin{cases} 2\mu - \sigma - \frac{1}{2} & \text{for } \mu > 1, \\ \frac{3}{2}\mu - \sigma & \text{for } 0 \leq \mu \leq 1 \end{cases}$$

“Philosophy” of the proof.

1. First establish continuity of the mapping $u \rightarrow \tilde{u}$, where $\tilde{u}(\theta) = u(\cos(\theta))$, from the weighted Sobolev space $H^m_\omega(I)$ into the corresponding periodic Sobolev space $H^m_p(-\pi, \pi)$
2. Then leverage analogous approximation error bounds established for the case of trigonometric basis functions

- ▶ Consider an approximation of $u \in L^2_\omega(I)$ in terms of a truncated Chebyshev series (expansion coefficients as the unknowns)

$$u_N(x) = \sum_{k=0}^N \hat{u}_k T_k(x)$$

- ▶ Cancel the residual $R_N = u - u_N$ on the set of GAUSS-LOBATTO-Chebyshev collocation points x_j , $j = 0, \dots, N$ (one could choose other sets of collocation points as well)

$$u(x_j) = \sum_{k=0}^N \hat{u}_k T_k(x_j), \quad j = 0, \dots, N$$

- ▶ Noting that $T_k(x_j) = \cos\left(k \cos^{-1}\left(\cos\left(\frac{j\pi}{N}\right)\right)\right) = \cos\left(k \frac{j\pi}{N}\right)$ and denoting $u_j \triangleq u(x_j)$ we obtain

$$u_j = \sum_{k=0}^N \hat{u}_k \cos\left(k \frac{\pi j}{N}\right), \quad j = 0, \dots, N$$

- ▶ The above system of equations can be written as $U = \mathcal{T}\hat{U}$, where U and \hat{U} are vectors of grid values and expansion coefficients, respectively.

- ▶ In fact, the matrix \mathcal{T} is invertible and

$$[\mathcal{T}^{-1}]_{jk} = \frac{2}{\bar{c}_j \bar{c}_k N} \cos\left(\frac{k\pi j}{N}\right), \quad j, k = 0, \dots, N$$

- ▶ Consequently, the expansion coefficients can be expressed as follows

$$\hat{u}_k = \frac{2}{\bar{c}_k N} \sum_{j=0}^N \frac{1}{\bar{c}_j} u_j \cos\left(\frac{k\pi j}{N}\right) = \frac{2}{\bar{c}_k N} \sum_{j=0}^N \frac{1}{\bar{c}_j} u_j \Re\left[e^{i\left(\frac{k\pi j}{N}\right)}\right], \quad k = 0, \dots, N$$

Note that this expression is nothing else than the **COSINE TRANSFORM** of U which can be very efficiently evaluated using a **COSINE FFT**

- ▶ The same expression can be obtained by
 - ▶ multiplying each side of $u_j = \sum_{k=0}^N \hat{u}_k T_k(x_j)$ by $\frac{T_l(x_j)}{\bar{c}_j}$
 - ▶ summing the resulting expression from $j = 0$ to $j = N$
 - ▶ using the **DISCRETE ORTHOGONALITY RELATION**

$$\frac{\pi}{N} \sum_{j=0}^N \frac{1}{\bar{c}_j} T_k(\tilde{\xi}_j) T_l(\tilde{\xi}_j) = \frac{\pi \bar{c}_k}{2} \delta_{kl}$$

- ▶ Note that the expression for the **DISCRETE CHEBYSHEV TRANSFORM**

$$\hat{u}_k = \frac{2}{c_k N} \sum_{j=0}^N \frac{1}{c_j} u_j \cos\left(\frac{k\pi j}{N}\right), \quad k = 0, \dots, N$$

can also be obtained by using the **Gauss-Lobatto-Chebyshev** quadrature to approximate the continuous expressions

$$\hat{u}_k = \frac{2}{\pi c_k} \int_{-1}^1 u T_k \omega dx, \quad k = 0, \dots, N,$$

Such an approximation is **EXACT** for $u \in \mathbb{P}_N$

- ▶ Analogous expressions for the Discrete Chebyshev Transforms can be derived for other set of collocation points (Gauss, Gauss-Radau)
- ▶ Note similarities with respect to the case periodic functions and the Discrete Fourier Transform

- ▶ As was the case with Fourier spectral methods, there is a very close connection between **COLLOCATION-BASED INTERPOLATION** and **GALERKIN APPROXIMATION**
- ▶ **DISCRETE CHEBYSHEV TRANSFORM** can be associated with an **INTERPOLATION OPERATOR** $P_C : C^0(I) \rightarrow \mathbb{R}^N$ defined such that $(P_C u)(x_j) = u(x_j)$, $j = 0, \dots, N$ (where x_j are the Gauss-Lobatto collocation points)

Theorem

Let $s > \frac{1}{2}$ and σ be given and $0 \leq \sigma \leq s$. There exists a constant C such that

$$\|u - P_C u\|_{\sigma, \omega} < CN^{2\sigma-s} \|u\|_{s, \omega}$$

for all $u \in H_\omega^s(I)$.

Outline of the Proof.

Changing the variables to $\tilde{u}(\theta) = u(\cos(\theta))$ we convert this problem to a problem already analyzed in the context of the Fourier interpolation for periodic functions □

- Relation between the **GALERKIN** and **COLLOCATION** coefficients, i.e.,

$$\hat{u}_k^e = \frac{2}{\pi c_k} \int_{-1}^1 u(x) T_k(x) \omega(x) dx, \quad k = 0, \dots, N$$

$$\hat{u}_k^c = \frac{2}{\bar{c}_k N} \sum_{j=0}^N \frac{1}{\bar{c}_j} u_j \cos\left(\frac{k\pi j}{N}\right), \quad k = 0, \dots, N$$

- Using the representation $u(x) = \sum_{l=0}^{\infty} \hat{u}_l^e T_l(x)$ in the latter expression and invoking the discrete orthogonality relation we obtain

$$\begin{aligned} \hat{u}_k^c &= \frac{2}{\bar{c}_k N} \sum_{l=0}^N \hat{u}_l^e \left[\sum_{j=0}^N \frac{1}{\bar{c}_j} T_k(x_j) T_l(x_j) \right] + \frac{2}{\bar{c}_k N} \sum_{l=N+1}^{\infty} \hat{u}_l^e \left[\sum_{j=0}^N \frac{1}{\bar{c}_j} T_k(x_j) T_l(x_j) \right], \\ &= \hat{u}_k^e + \frac{2}{\bar{c}_k N} \sum_{l=N+1}^{\infty} \hat{u}_l^e C_{kl} \end{aligned}$$

$$\text{where } C_{kl} = \sum_{j=0}^N \frac{1}{\bar{c}_j} T_k(x_j) T_l(x_j) = \sum_{j=0}^N \frac{1}{\bar{c}_j} \cos\left(\frac{kj\pi}{N}\right) \cos\left(\frac{lj\pi}{N}\right)$$

$$= \frac{1}{2} \sum_{j=0}^N \frac{1}{\bar{c}_j} \left[\cos\left(\frac{k-l}{N} j\pi\right) + \cos\left(\frac{k+l}{N} j\pi\right) \right]$$

- ▶ Using the identity

$$\sum_{j=0}^N \cos\left(\frac{pj\pi}{N}\right) = \begin{cases} N+1, & \text{if } p = 2mN, \quad m = 0, \pm 1, \pm 2, \dots \\ \frac{1}{2}[1 + (-1)^p] & \text{otherwise} \end{cases}$$

we can calculate C_{kl} which allows us to express the relation between the Galerkin and collocation coefficients as follows

$$\hat{u}_k^c = \hat{u}_k^e + \frac{1}{\bar{c}_k} \left[\sum_{\substack{m=1 \\ 2mN > N-k}}^{\infty} \hat{u}_{k+2mN}^e + \sum_{\substack{m=1 \\ 2mN > N+k}}^{\infty} \hat{u}_{-k+2mN}^e \right]$$

- ▶ The terms in square brackets represent the **ALIASING ERRORS**. Their origin is precisely the same as in the Fourier (pseudo)-spectral method.
- ▶ Aliasing errors can be removed using the **3/2 APPROACH** in the same way as in the Fourier (pseudo)-spectral method

- expressing the first N Chebyshev polynomials as functions of x^k ,
 $k = 1, \dots, N$

$$T_0(x) = 1,$$

$$T_1(x) = x,$$

$$T_2(x) = 2x^2 - 1,$$

$$T_3(x) = 4x^3 - 3x,$$

$$T_4(x) = 8x^4 - 8x^2 + 1$$

which can be written as $V = \mathbb{K}X$, where $[V]_k = T_k(x)$, $[X]_k = x^k$,
 and \mathbb{K} is a **LOWER-TRIANGULAR** matrix

- Solving this system (trivially!) results in the following **RECIPROCAL RELATIONS**

$$1 = T_0(x),$$

$$x = T_1(x),$$

$$x^2 = \frac{1}{2}[T_0(x) + T_2(x)],$$

$$x^3 = \frac{1}{4}[3T_1(x) + T_3(x)],$$

$$x^4 = \frac{1}{8}[3T_0(x) + 4T_2(x) + T_4(x)]$$

- ▶ Find the best polynomial approximation of order 3 of $f(x) = e^x$ on $[-1, 1]$
- ▶ Construct the (Maclaurin) expansion

$$e^x = 1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3 + \frac{1}{24}x^4 + \dots$$

- ▶ Rewrite the expansion in terms of **CHEBYSHEV POLYNOMIALS** using the reciprocal relations

$$e^x = \frac{81}{64} T_0(x) + \frac{9}{8} T_1(x) + \frac{13}{48} T_2(x) + \frac{1}{24} T_3(x) + \frac{1}{192} T_4(x) + \dots$$

- ▶ Truncate this expansion to the 3rd order and translate the expansion back to the x^k representation
- ▶ Truncation error is given by the magnitude of the first truncated term; Note that the **CHEBYSHEV EXPANSION COEFFICIENTS** are much smaller than the corresponding **TAYLOR EXPANSION COEFFICIENTS** !
- ▶ How is it possible – the same number of expansion terms, but higher accuracy?

- ▶ Assume function approximation in the form $u_N(x) = \sum_{k=0}^N \hat{u}_k T_k(x)$
- ▶ First, note that CHEBYSHEV PROJECTION and DIFFERENTIATION do not commute, i.e., $P_N(\frac{du}{dx}) \neq \frac{d}{dx}(P_N u)$
- ▶ Sequentially applying the recurrence relation $2T_k = \frac{T'_{k+1}}{k+1} - \frac{T'_{k-1}}{k-1}$ we obtain

$$T'_k(x) = 2k \sum_{p=0}^K \frac{1}{c_{k-1-2p}} T_{k-1-2p}(x), \quad \text{where } K = \left\lfloor \frac{k-1}{2} \right\rfloor$$

- ▶ Consider the first derivative

$$u'_N(x) = \sum_{k=0}^N \hat{u}_k T'_k(x) = \sum_{k=0}^N \hat{u}_k^{(1)} T_k(x)$$

where, using the above expression for $T'_k(x)$, we obtain the expansion coefficients as

$$\hat{u}_k^{(1)} = \frac{2}{c_k} \sum_{\substack{p=k+1 \\ (p+k) \text{ odd}}}^N p \hat{u}_p, \quad k = 0, \dots, N-1, \quad \text{and} \quad \hat{u}_N^{(1)} = 0$$

- ▶ Spectral differentiation (with the expansion coefficients as unknowns) can thus be written as

$$\hat{U}^{(1)} = \hat{\mathbb{D}}\hat{U},$$

where $\hat{U} = [\hat{u}_0 \dots, \hat{u}_N]^T$, $\hat{U}^{(1)} = [\hat{u}_0^{(1)} \dots, \hat{u}_N^{(1)}]^T$, and $\hat{\mathbb{D}}$ is an **UPPER-TRIANGULAR** matrix with entries deduced based on the previous expression

- ▶ For the second derivative one obtains similarly

$$u_N''(x) = \sum_{k=0}^N \hat{u}_k^{(2)} T_k(x)$$

$$\hat{u}_k^{(2)} = \frac{1}{c_k} \sum_{\substack{p=k+2 \\ (p+k) \text{ even}}}^N p(p^2 - k^2) \hat{u}_p, \quad k = 0, \dots, N-2$$

and $\hat{u}_N^{(2)} = \hat{u}_{N-1}^{(2)} = 0$

- ▶ **QUESTION** — What is the structure of the second-order differentiation matrix?

- Assume the function $u(x)$ is approximated in terms of its nodal values, i.e.,

$$u(x) \cong u_N(x) = \sum_{j=0}^N u(x_j) C_j(x),$$

where $\{x_j\}$ are the **GAUSS-LOBATTO-CHEBYSHEV** points and $C_j(x)$ are the associated **CARDINAL FUNCTIONS**

$$C_j(x) = (-1)^{j+1} \frac{(1-x^2)}{c_j N^2 (x-x_j)} \frac{dT_N(x)}{dx} = \frac{2}{N p_j} \sum_{m=0}^N \frac{1}{p_m} T_m(x_j) T_m(x),$$

where

$$p_j = \begin{cases} 2 & \text{for } j = 0, N, \\ 1 & \text{for } j = 1, \dots, N-1, \end{cases} \quad c_j = \begin{cases} 2 & \text{for } j = N, \\ 1 & \text{for } j = 0, \dots, N-1 \end{cases}$$

- The **DIFFERENTIATION MATRIX** $\mathbb{D}^{(p)}$ relating the nodal values of the p -th derivative $u_N^{(p)}$ to the nodal values of u is obtained by differentiating the cardinal function appropriate number of times

$$u_N^{(p)}(x_j) = \sum_{k=0}^N \frac{d^{(p)} C_k(x_j)}{dx^{(p)}} u(x_k) = \sum_{k=0}^N d_{jk}^{(p)} u(x_k), \quad j = 0, \dots, N$$

- Expressions for the entries of the **DIFFERENTIATION MATRIX** $d_{jk}^{(1)}$ at the the **GAUSS-LOBATTO-Chebyshev** collocation points

$$d_{jk}^{(1)} = \frac{\bar{c}_j}{\bar{c}_k} \frac{(-1)^{j+k}}{x_j - x_k}, \quad 0 \leq j, k \leq N, j \neq k,$$

$$d_{jj}^{(1)} = -\frac{x_j}{2(1-x_j^2)}, \quad 1 \leq j \leq N-1,$$

$$d_{00}^{(1)} = -d_{NN}^{(1)} = \frac{2N^2 + 1}{6},$$

- Thus in the matrix (operator) notation

$$U^{(1)} = \mathbb{D}U$$

- Note that **Rows** of the differentiation matrix \mathbb{D} are in fact equivalent to N -th order asymmetric finite-difference formulas on a nonuniform grid; in other words, spectral differentiation using nodal values as unknowns is equivalent to finite differences employing **ALL N GRID POINTS AVAILABLE**

- Expressions for the entries of **SECOND-ORDER DIFFERENTIATION MATRIX** $d_{jk}^{(2)}$ at the the **GAUSS-LOBATTO-CHEBYSHEV** collocation points ($U^{(2)} = \mathbb{D}^{(2)} U$)

$$d_{jk}^{(2)} = \frac{(-1)^{j+k}}{\bar{c}_k} \frac{x_j^2 + x_j x_k - 2}{(1 - x_j^2)(x_j - x_k)^2}, \quad 1 \leq j \leq N-1, 0 \leq k \leq N, j \neq k$$

$$d_{jj}^{(2)} = -\frac{(N^2 - 1)(1 - x_j^2) + 3}{3(1 - x_j^2)^2}, \quad 1 \leq j \leq N-1,$$

$$d_{0k}^{(2)} = \frac{2}{3} \frac{(-1)^k}{\bar{c}_k} \frac{(2N^2 + 1)(1 - x_k) - 6}{(1 - x_k)^2}, \quad 1 \leq k \leq N$$

$$d_{Nk}^{(2)} = \frac{2}{3} \frac{(-1)^{N+k}}{\bar{c}_k} \frac{(2N^2 + 1)(1 + x_k) - 6}{(1 + x_k)^2}, \quad 0 \leq k \leq N-1$$

$$d_{00}^{(2)} = d_{NN}^{(2)} = \frac{N^4 - 1}{15},$$

- Note that
$$d_{jk}^{(2)} = \sum_{p=0}^N d_{jp}^{(1)} d_{pk}^{(1)}$$

- Interestingly, \mathbb{D}^2 is not a **SYMMETRIC MATRIX** ...

- Consider an **ELLIPTIC BOUNDARY VALUE PROBLEM (BVP)** :

$$\begin{aligned} -\nu u'' + au' + bu &= f, & \text{in } [-1, 1] \\ \alpha_- u + \beta_- u' &= g_- & x = -1 \\ \alpha_+ u + \beta_+ u' &= g_+ & x = 1 \end{aligned}$$

- Chebyshev polynomials do not satisfy homogeneous boundary conditions, hence standard Galerkin approach is not directly applicable.
- **BASIS RECOMBINATION** :

- Convert the BVP to the corresponding form with **HOMOGENEOUS BOUNDARY CONDITIONS**
- Take linear combinations of Chebyshev polynomials to construct a new basis satisfying **HOMOGENEOUS DIRICHLET BOUNDARY CONDITIONS**
 $\varphi_k(\pm 1) = 0$

$$\varphi_k(x) = \begin{cases} T_k(x) - T_0(x) = T_k - 1, & k - \text{even} \\ T_k(x) - T_1(x), & k - \text{odd} \end{cases}$$

Note that the new basis preserves orthogonality

- ▶ **THE TAU METHOD** (Lanczos, 1938) consists in using a Galerkin approach in which explicit enforcement of the boundary conditions replaces projections on some of the test functions
- ▶ Consider the residual

$$R_N(x) = -\nu u_N'' + au_N' + bu_N - f,$$

where $u_N(x) = \sum_{k=0}^N \hat{u}_k T_k(x)$

- ▶ Cancel projections of the residual on the first $N - 2$ basis functions

$$(R_N, T_l)_\omega = \sum_{k=0}^N \left(-\nu \hat{u}_k^{(2)} + a \hat{u}_k^{(1)} + b \hat{u}_k \right) \int_{-1}^1 T_k T_l \omega dx - \int_{-1}^1 f T_l \omega dx, \quad l = 0, \dots, N - 2$$

- ▶ Thus, using orthogonality, we obtain

$$-\nu \hat{u}_k^{(2)} + a \hat{u}_k^{(1)} + b \hat{u}_k = \hat{f}_k, \quad k = 0, \dots, N - 2$$

where $\hat{f}_k = \int_{-1}^1 f T_k \omega dx$

- ▶ Noting that $T_k(\pm 1) = (\pm 1)^k$ and $T'_k(\pm 1) = (\pm 1)^{k+1}k^2$, the **BOUNDARY CONDITIONS** are enforced by supplementing the residual equations with

$$\sum_{k=0}^N (-1)^k (\alpha_- - \beta_- k^2) \hat{u}_k = g_-$$

$$\sum_{k=0}^N (-1)^k (\alpha_+ + \beta_+ k^2) \hat{u}_k = g_+$$

- ▶ Expressing $\hat{u}_k^{(2)}$ and $\hat{u}_k^{(1)}$ in terms of \hat{u}_k via the Chebyshev spectral differentiation matrices we obtain the following system

$$\hat{\mathbb{A}} \hat{\mathbb{U}} = \hat{\mathbb{F}}$$

where $\hat{\mathbb{U}} = [\hat{u}_0, \dots, \hat{u}_N]^T$, $\hat{\mathbb{F}} = [\hat{f}_0, \dots, \hat{f}_{N-2}, \mathbf{g}_-, \mathbf{g}_+]$ and the matrix $\hat{\mathbb{A}}$ is obtained by adding the two rows representing the boundary conditions (see above) to the matrix $\hat{\mathbb{A}}_1 = -\nu \hat{\mathbb{D}}^2 + a \hat{\mathbb{D}} + bI$.

- ▶ When the domain boundary is not just a point (e.g., in 2D / 3D), formulation of the Tau method becomes somewhat more involved

- ▶ Consider the residual

$$R_N(x) = -\nu u_N'' + au_N' + bu_N - f_N,$$

where $u_N(x) = \sum_{k=0}^N \hat{u}_k T_k(x)$, $f(x) \approx f_N(x) = \sum_{k=0}^N \hat{f}_k T_k(x)$

- ▶ Cancel this residual at $N - 1$ GAUSS-LOBATTO-CHEBYSHEV collocation points located in the interior of the domain

$$-\nu u_N''(x_j) + au_N'(x_j) + bu_N(x_j) = f_N(x_j), \quad j = 1, \dots, N - 1$$

- ▶ Enforce the two boundary conditions at endpoints

$$\alpha_- u_N(x_N) + \beta_- u_N'(x_N) = g_-$$

$$\alpha_+ u_N(x_0) + \beta_+ u_N'(x_0) = g_+$$

Note that this shows the utility of using the GAUSS-LOBATTO-CHEBYSHEV collocation points

- Consequently, the following system of $N + 1$ equations is obtained

$$\sum_{k=0}^N (-\nu d_{jk}^{(2)} + a d_{jk}^{(1)}) \hat{u}_k T_k(x_j) + b \sum_{k=0}^N \hat{u}_k T_k(x_j) = \sum_{k=0}^N \hat{f}_k T_k(x_j), \quad j = 1, \dots, N - 1$$

$$\alpha_- \sum_{k=0}^N \hat{u}_k T_k(x_N) + \beta_- \sum_{k=0}^N d_{Nk}^{(1)} \hat{u}_k T_k(x_k) = g_-$$

$$\alpha_+ \sum_{k=0}^N \hat{u}_k T_k(x_0) + \beta_+ \sum_{k=0}^N d_{0k}^{(1)} \hat{u}_k T_k(x_k) = g_+$$

which can be written as $\hat{\mathbb{A}}_c \hat{\mathbb{U}} = \hat{\mathbb{F}}$, where $[\hat{\mathbb{A}}_c]_{jk} = [\hat{\mathbb{A}}_{c0}]_{jk}$, $j, k = 1, \dots, N - 1$ with $\hat{\mathbb{A}}_{c0}$ given by

$$\hat{\mathbb{A}}_{c0} = (-\nu \hat{\mathbb{D}}^2 + a \hat{\mathbb{D}} + b \mathbb{I})$$

and the **BOUNDARY CONDITIONS** above added as the rows **0** and **N** of $\hat{\mathbb{A}}_c$

- ▶ Note that the matrix corresponding to this system of equations may be **POORLY CONDITIONED** , so special care must be exercised when solving this system for large N .
- ▶ Similar approach can be used when the nodal values $u(x_j)$, rather than the Chebyshev coefficients \hat{u}_k are unknowns

- ▶ When the equation has **NONCONSTANT COEFFICIENTS**, similar difficulties as in the Fourier case are encountered (evaluation of **CONVOLUTION SUMS**)
- ▶ Consequently, the **COLLOCATION** (pseudo-spectral) approach is preferable along the guidelines laid out in the case of the Fourier spectral methods
- ▶ Assuming $a = a(x)$ in the elliptic boundary value problem, we need to make the following modification to \mathbb{A}_c :

$$\mathbb{A}'_{c0} = (-\nu \mathbb{D}^2 + \mathbb{D}' + b\mathbb{I})U,$$

where $\mathbb{D}' = [a(x_j)d_{jk}^{(1)}]$, $j, k = 1, \dots, N$

- ▶ For the Burgers equation $\partial_t u + \frac{1}{2}\partial_x u^2 - \nu \partial_x^2 u$ we obtain at every time step n

$$(\mathbb{I} - \Delta t \nu \mathbb{D}^{(2)})U^{n+1} = U^n - \frac{1}{2}\Delta t \mathbb{D} W^n,$$

where $[W^n]_j = [U^n]_j[U^n]_j$; Note that an algebraic system has to be solved at each time step

Epilogue — Domain Decomposition

► Motivation:

- treatment of problem in **IRREGULAR DOMAINS**
- **STIFF PROBLEMS**

► **PHILOSOPHY** — partition the original domain Ω into a number of **SUBDOMAINS** $\{\Omega_m\}_{m=1}^M$ and solve the problem separately on each those while respecting consistency conditions on the interfaces

► **Spectral Element Method**

- consider a collection of problems posed on each subdomain Ω_m

$$\mathcal{L}u_m = f$$

$$u_{m-1}(a_m) = u_m(a_m), \quad u_m(a_{m+1}) = u_{m+1}(a_{m+1})$$

- Transform each subdomain Ω_m to $I = [-1, 1]$
- use a separate set of N_m **ORTHOGONAL POLYNOMIALS** to approximate the solution on every subinterval
- boundary conditions on interfaces provide coupling between problems on subdomains

