

**WELCOME TO MATH 745 –
– TOPICS IN NUMERICAL ANALYSIS**

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PART I

Review of Interpolation and Approximation Theory

*Although this may seem a paradox,
all exact science is dominated by the idea of approximation.*

— Bertrand Russell (1872–1970)

REVIEW OF INTERPOLATION AND APPROXIMATION THEORY I

- Goal — given a function f , known exactly or only approximately, find its representation that has
 - a more simply computable form, and
 - error of interpolation / approximation is within some prescribed tolerance.
- Interpolation vs. Approximation
- Reduction to **an algebraic problem** :
 - existence of a solution,
 - its uniqueness,
 - conditioning of the resulting algebraic system
- Error estimates

INTERPOLATION THEORY I

ABSTRACT FRAMEWORK

- Let:
 - V be a **normed vector space** over a field \mathbb{K} of numbers (real \mathbb{R} , or complex \mathbb{C})
 - V' be the **dual space**, i.e., a space of **linear bounded functionals** on V
- Statement of an abstract INTERPOLATION PROBLEM:
 - suppose $V_n \in V$ is an n -dimensional subspace of V with a basis $\{v_1, \dots, v_n\}$
 - let $L_i \in V'$, $1 \leq i \leq n$ be n linear functionals
 - given n numbers $b_i \in \mathbb{R}$, $1 \leq i \leq n$, find $u_n \in V_n$ such that the following **interpolation conditions** are satisfied:

$$L_i u_n = b_i, \quad 1 \leq i \leq n$$

- Questions:
 - Does the interpolation problem have a solutions?
 - If so, is this solution unique?
 - Can we estimate the error?

INTERPOLATION THEORY II ABSTRACT FRAMEWORK

- Definition — The functionals L_i , $1 \leq i \leq n$, are **linearly independent** over V_n if

$$\forall v \in V_n, \sum_{i=1}^n \alpha_i L_i(v) = 0 \implies \alpha_i = 0, \quad 1 \leq i \leq n$$

- Lemma — the linear functionals L_1, \dots, L_n are linearly independent over V_n iff

$$\det(L_i v_j) = \det \begin{bmatrix} L_1 v_1 & \cdots & L_1 v_n \\ \vdots & \ddots & \vdots \\ L_n v_1 & \cdots & L_n v_n \end{bmatrix} \neq 0$$

- *Proof* — L_1, \dots, L_n linearly independent over V_n

$$\iff \sum_{i=1}^n \alpha_i L_i(v_j) = 0, \quad 1 \leq j \leq n \implies \alpha_i = 0, \quad 1 \leq i \leq n$$

$$\iff \det(L_i v_j) \neq 0.$$

INTERPOLATION THEORY III ABSTRACT FRAMEWORK

- Theorem — the following statements are equivalent:
 1. The interpolation problem has a unique solution
 2. The functionals L_1, \dots, L_n are linearly independent over V_n
 3. The only element $u_n \in V_n$ satisfying $L_i u_n = 0, 1 \leq i \leq n$ is $u_n = 0$
 4. For any data $\{b_i\}_{i=1}^n$, there exists one $u_n \in V_n$ such that

$$L_i u_n = b_i, \quad 1 \leq i \leq n$$

- *Proof* — from linear algebra, for a square matrix $\mathbb{A} \in \mathbb{K}^{n \times n}$, the following statements are equivalent:
 1. The system $\mathbb{A}\mathbf{x} = \mathbf{b}$ has a unique solution $\mathbf{x} \in \mathbb{K}^n$ for any $\mathbf{b} \in \mathbb{K}^n$,
 2. $\det(\mathbb{A}) \neq 0$,
 3. If $\mathbb{A}\mathbf{x} = \mathbf{0}$, the $\mathbf{x} = \mathbf{0}$,
 4. For any $\mathbf{b} \in \mathbb{K}^n$, the system $\mathbb{A}\mathbf{x} = \mathbf{b}$ has a solution $\mathbf{x} \in \mathbb{K}^n$.

The results of the theorem follow from the above statements.

INTERPOLATION THEORY IV ABSTRACT FRAMEWORK

- Given $u \in V$, its interpolant $u_n = \sum_{i=1}^n \alpha_i v_i$ in V_n is defined by the interpolation conditions

$$L_i u_n = L_i u, \quad 1 \leq i \leq n$$

- The coefficients $\{\alpha_i\}_{i=1}^n$ can be found from the linear system

$$\begin{bmatrix} L_1 v_1 & \cdots & L_1 v_n \\ \vdots & \ddots & \vdots \\ L_n v_1 & \cdots & L_n v_n \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \vdots \\ \alpha_n \end{bmatrix} = \begin{bmatrix} L_1 u \\ \vdots \\ L_n u \end{bmatrix}$$

which has a unique solution if the functionals L_1, \dots, L_n are linearly independent over V_n .

- Specific examples of interpolation:
 - Lagrange polynomial interpolation
 - Hermite polynomial interpolation
 - Piecewise polynomial interpolation
 - trigonometric interpolation

LAGRANGE POLYNOMIAL INTERPOLATION I

- Let:
 - $f : [a, b] \rightarrow \mathbb{K}$ be a continuous function defined on a finite closed interval $[a, b]$, so that one can choose $V = C[a, b]$
 - $\Delta : a \leq x_0 \leq x_1 \leq \cdots \leq x_n \leq b$ be a partition of the interval $[a, b]$
 - V_{n+1} be given by \mathcal{P}_n , the space of polynomials of degree less than or equal to n
- Defining the **interpolation linear functionals** as $L_i f = f(x_i)$, $0 \leq i \leq n$, we obtain the following conditions for the **Lagrange polynomial interpolant** of degree n

$$p_n(x_i) = f(x_i), \quad 0 \leq i \leq n, \quad p_n \in \mathcal{P}_n$$

LAGRANGE POLYNOMIAL INTERPOLATION II EXISTENCE OF SOLUTIONS AND UNIQUENESS

- Choosing the set of basis functions for \mathcal{P}_n as $v_j(x) = x^j$, $0 \leq j \leq n$, we obtain $L_i v_j$ in the form of the **Vandermonde matrix** :

$$L_i v_j = \begin{bmatrix} 1 & x_0 & x_0^2 & \dots & x_0^n \\ 1 & x_1 & x_1^2 & \dots & x_1^n \\ \vdots & \vdots & \vdots & \dots & \vdots \\ 1 & x_{n-1} & x_{n-1}^2 & \dots & x_{n-1}^n \\ 1 & x_n & x_n^2 & \dots & x_n^n \end{bmatrix}$$

The determinant of this matrix can be shown to be

$$\det(L_i v_j) = \prod_{j>i}^n (x_j - x_i) \neq 0.$$

Hence, there exists a **unique** Lagrange interpolation polynomial.

- **Caveat** — Note that the Vandermonde matrix is extremely ill-conditioned! This means that in the resulting algebraic system is hard to solve accurately and therefore in practice other methods are preferred.

LAGRANGE POLYNOMIAL INTERPOLATION III

- An alternative approach — **Lagrange's formula** for the interpolation polynomial:

$$p_n(x) = \sum_{i=0}^n f(x_i)\phi_i(x),$$

where $\{\phi_i\}_{i=0}^n$ are the **Lagrange basis functions** defined as

$$\phi_i(x) = \prod_{j \neq i} \frac{x - x_j}{x_i - x_j},$$

- Note that $\{\phi_i\}_{i=0}^n$ satisfy the interpolation condition

$$\phi_i(x_j) = \delta_{ij} = \begin{cases} 0, & i \neq j, \\ 1, & i = j, \end{cases}$$

and constitute a basis for the space \mathcal{P}_n .

- Solution obtained directly (i.e., without solving an algebraic system); Note that for homogeneous data the interpolant is zero which by the above theorem ensures uniqueness of the solution (this is often a very convenient way of proving existence and uniqueness).

LAGRANGE POLYNOMIAL INTERPOLATION IV ERROR ESTIMATES

- Theorem — Assume $f \in C^{(n+1)}[a, b]$. Then there exists a ξ_x between $\min(x_i, x)$ and $\max(x_i, x)$ such that the **local interpolation error** is

$$f(x) - p_n(x) = \frac{\omega_n(x)}{(n+1)!} f^{(n+1)}(\xi_x), \quad \text{where } \omega_n(x) = \prod_{i=0}^n (x - x_i)$$

- *Proof:*

- *The result is obvious for $x = x_i$, $0 \leq i \leq n$,*
- *Suppose $x \neq x_i$, $0 \leq i \leq n$, and denote $E(x) = f(x) - p_n(x)$.*

Consider the function:
$$g(t) = E(t) - \frac{\omega_n(t)}{\omega_n(x)} E(x)$$

We note that $g(t)$ has $(n+2)$ distinct roots: $t = x$ and $t = x_i$, $0 \leq i \leq n$.

By the Mean Value Theorem, $g'(t)$ has $(n+1)$ distinct roots. Repeatedly applying the Mean Value Theorem to derivatives of g we conclude that $g^{(n+1)}(t)$ has a root $\xi_x \in (\min(x_i, x), \max(x_i, x))$. Thus

$$g^{(n+1)}(t) = f^{(n+1)}(\xi_x) - \frac{(n+1)!}{\omega_n(x)} E(x) = 0 \quad \blacksquare$$

LAGRANGE POLYNOMIAL INTERPOLATION V REMARKS

- **Runge Phenomenon** — Examination of the interpolation error formula shows that there may be points x^* in the interval $[a, b]$ such that

$$\lim_{n \rightarrow \infty} |f(x^*) - p_n(x^*)| = \infty,$$

meaning that the sequence of the approximation polynomials $p_n(x)$ does not uniformly converge to f !

Such situation often arises close to the endpoints of the interpolation interval $[a, b]$.

- This difficulty can be mitigated by performing interpolation on sets of **non-uniformly** spaced grid points — for instance, zeros of Chebyshev polynomials

HERMITE POLYNOMIAL INTERPOLATION I

- General idea — use the values of both $f(x)$ and $f'(x)$ as interpolation conditions
- Assume that $f \in C^1[a, b]$ and let $\Delta : a \leq x_1 \leq \dots \leq x_n \leq b$ be a partition of the interval $[a, b]$
- The **Hermite interpolation polynomial** $p_{2n-1} \in \mathcal{P}_{2n-1}$ is chosen to satisfy the conditions

$$p_{2n-1}(x_i) = f(x_i), \quad p'_{2n-1}(x_i) = f'(x_i), \quad 1 \leq i \leq n$$

- More generally, given a set of integers $\{m_i\}_{i=1}^n$ and $f \in C^M[a, b]$, where $M = \max_i(m_i)$, the Hermite interpolation problem can be stated as follows — find an $p_N \in \mathcal{P}_N$, $N = \sum_{i=1}^n (m_i + 1) - 1$ such that

$$p_N^{(j)}(x_i) = f^{(j)}(x_i), \quad 0 \leq j \leq m_i, \quad 1 \leq i \leq n$$

- Similar results concerning existence and uniqueness of solutions and the corresponding error bounds can be proven as for the Lagrange interpolation.

PIECEWISE POLYNOMIAL INTERPOLATION I

- Assume that $f \in C[a, b]$ and let $\Delta : a \leq x_0 \leq x_1 \leq \dots \leq x_n \leq b$ be a partition of the interval $[a, b]$; denote $h_i = x_i - x_{i-1}$, $1 \leq i \leq n$ and $h = \max_{1 \leq i \leq n} h_i$.
- The piecewise linear interpolant is defined as follows:
 - for each $i = 1, \dots, n$ $\Pi_{\Delta} f|_{[x_{i-1}, x_i]}$ is linear,
 - for $i = 0, 1, \dots, n$, $\Pi_{\Delta} f(x_i) = f(x_i)$
- It is easy to see that the following unique linear interpolation polynomial exists:

$$\Pi_{\Delta} f(x) = \frac{x_i - x}{h_i} f(x_{i-1}) + \frac{x - x_{i-1}}{h_i} f(x_i), \quad x \in [x_{i-1}, x_i]$$

- Suppose that $f \in C^2[a, b]$; **local** interpolation error can be assessed using the estimate derived previously, i.e.

$$\max_{x \in [a, b]} |f(x) - \Pi_{\Delta} f(x)| \leq \frac{h^2}{8} \max_{x \in [a, b]} |f''(x)|$$

- What about **global error** (i.e., in the integral sense)?

PIECEWISE POLYNOMIAL INTERPOLATION II

- Instead of $f \in C^2[a, b]$, assume now that $f \in H^2(a, b)$
- Consider the interpolation error in the L_2 sense

$$\|f(x) - \Pi_{\Delta} f(x)\|_{L^2(a,b)}^2 = \int_a^b |f(x) - \Pi_{\Delta} f(x)|^2 dx = \sum_{i=1}^n \int_{x_{i-1}}^{x_i} |f(x) - \Pi_{\Delta} f(x)|^2 dx$$

- Introduce a function $\hat{f} \in H^2(0, 1)$ with its linear interpolant

$$\hat{\Pi}\hat{f}(\xi) = \hat{f}(0)(1 - \xi) + \hat{f}(1)\xi, \quad 0 \leq \xi \leq 1$$

- By Taylor's theorem

$$\hat{f}(0) = \hat{f}(\xi) - \xi \hat{f}'(\xi) - \int_{\xi}^0 t \hat{f}''(t) dt$$

$$\hat{f}(1) = \hat{f}(\xi) + (1 - \xi) \hat{f}'(\xi) + \int_{\xi}^1 (1 - t) \hat{f}''(t) dt, \quad \text{then}$$

$$|\hat{f}(\xi) - \hat{\Pi}\hat{f}(\xi)| = \left| \xi \int_{\xi}^1 (1 - t) \hat{f}''(t) dt + (1 - \xi) \int_0^{\xi} t \hat{f}''(t) dt \right| \leq \xi(1 - \xi) \left| \int_0^1 \hat{f}''(\xi) d\xi \right|$$

and therefore (by Cauchy's theorem)

$$\int_0^1 |\hat{f}(\xi) - \hat{\Pi}\hat{f}(\xi)|^2 d\xi \leq c \int_0^1 |\hat{f}''(\xi)|^2 d\xi$$

for some c independent of \hat{f} .

PIECEWISE POLYNOMIAL INTERPOLATION III

- On the other hand

$$\begin{aligned} \int_{x_{i-1}}^{x_i} |f(x) - \Pi_{\Delta} f(x)|^2 dx &= h_i \int_0^1 |f(x_{i-1} + h_i \xi) - \hat{\Pi} f(x_{i-1} + h_i \xi)|^2 d\xi \\ &\leq ch_i \int_0^1 \left| \frac{d^2 f(x_{i-1} + h_i \xi)}{d\xi^2} \right|^2 d\xi \\ &\leq ch_i^5 \int_0^1 |f''(x_{i-1} + h_i \xi)|^2 d\xi = ch_i^4 \int_{x_{i-1}}^{x_i} |f''(x)|^2 dx \end{aligned}$$

- Therefore

$$\|f(x) - \Pi_{\Delta} f(x)\|_{L^2(a,b)}^2 = \sum_{i=1}^n \int_{x_{i-1}}^{x_i} |f(x) - \Pi_{\Delta} f(x)|^2 dx \leq ch^4 \|f''\|_{L^2(a,b)}^2,$$

or
$$\|f(x) - \Pi_{\Delta} f(x)\|_{L^2(a,b)} \leq ch^2 \|f''\|_{L^2(a,b)}$$

- A similar argument shows that

$$\|f'(x) - (\Pi_{\Delta} f(x))'\|_{L^2(a,b)}^2 \leq c_1 h \|f''\|_{L^2(a,b)}$$

TRIGONOMETRIC INTERPOLATION I

- When considering periodic functions, an important class of interpolating functions is given by **trigonometric polynomials**

$$p_n(x) = a_0 + \sum_{j=1}^n [a_j \cos(jx) + b_j \sin(jx)]$$

When $|a_n| + |b_n| \neq 0$ $p_n(x)$ is said to be of order n .

- The polynomial can equivalently be written as

$$p_n(x) = \sum_{j=-n}^n c_j e^{ijx} = \sum_{j=-n}^n c_j z^j = z^{-n} \sum_{k=0}^{2n} c_{k-n} z^k,$$

where $a_0 = c_0$, $a_j = c_j + c_{-j}$, $b_j = i(c_j - c_{-j})$, and $z = e^{ix}$. Note similarity of the last expression to the algebraic polynomials described before.

TRIGONOMETRIC INTERPOLATION II

- Consider a periodic domain $[0, 2\pi)$ with the following partition
 $\Delta : 0 \leq x_0 \leq x_1 \leq \dots \leq x_{2n} < 2\pi$, where $x_j = jh$, $j = 0, 1, \dots, 2n$ and
 $h = \frac{2\pi}{2n+1}$
- Given the numbers $\{b_j\}_{j=0}^{2n}$ and introducing the complex nodes $z_j = e^{ix_j}$ ($j = 0, 1, \dots, 2n$), the trigonometric interpolation problem can be stated as follows

$$\sum_{k=0}^{2n} c_{k-n} z_j^k = z_j^n b_j, \quad j = 0, 1, \dots, 2n$$

- Existence and uniqueness of a solution to this problem can be proven by reducing it to the solution of the complex Lagrange interpolation problem.

APPROXIMATION — ABSTRACT FRAMEWORK I

- The **best approximation problem** consists in choosing the member of a restricted class of functions that provides the best representation of the given function f when the approximation error is measured in a prescribed norm; Solution of this problem depends on:

- properties of the function f
- properties of the set of approximating functions
- properties of the norm quantifying the approximation error

- More precisely, given a normed vector space V with some $u \in V$ and a set $K \subseteq V$, we are interested in solution of the following minimization problem

$$\inf_{v \in K} \|u - v\|$$

- Remark — recall analogy with the classical theorem by Weierstraß:
 - Theorem — a real-valued **continuous** function $f : \mathbb{R} \rightarrow \mathbb{R}$ on a **bounded closed** interval $[a, b]$ ($-\infty < a < b < \infty$) has a maximum and a minimum.

APPROXIMATION — ABSTRACT FRAMEWORK II

- Depending on the setting, existence of a best approximation is addressed by the following two theorems:
 - Theorem — Assume $K \subseteq V$ is a **convex and closed finite-dimensional** subset of a normed space V . Then there is an element $\hat{u} \in K$ such that
$$\|u - \hat{u}\| = \inf_{v \in K} \|u - v\|$$
 - Theorem — Assume K is a **finite-dimensional subspace** of a normed space V . Then there is an element $\hat{u} \in K$ such that
$$\|u - \hat{u}\| = \inf_{v \in K} \|u - v\|$$
- Let $V = L^p(a, b)$ and $K = \mathcal{P}_n$, the space of all the polynomials of degree less than or equal to n . Associated with the space V , we may use $L^p(a, b)$ norms with $1 \leq p \leq \infty$. The above results ensure that for any $f \in L^p(a, b)$ there exists a polynomial $f_n \in \mathcal{P}_n$ such that

$$\|f - f_n\|_{L^p(a,b)} = \inf_{q_n \in \mathcal{P}_n} \|f - q_n\|_{L^p(a,b)}$$

For different values of p we have different best approximations. When $p = \infty$, f_n is called a **best uniform approximation** of f .

APPROXIMATION IN HILBERT SPACES I

- When V is an **inner-product** (Hilbert) space, the norm $\|\cdot\|$ is induced by the associated inner product; this significantly simplifies solution of an approximation problem.
- Definition — Let V be a linear space and V_1 and V_2 subspaces of V . We say that V is a **direct sum** of V_1 and V_2 and write $V = V_1 \oplus V_2$ if any element $v \in V$ can be uniquely decomposed as

$$v = v_1 + v_2, \quad v_1 \in V_1, \quad v_2 \in V_2.$$

Furthermore, if V is an **inner-product** space and $(v_1, v_2) = 0$ for any $v_1 \in V_1$ and $v_2 \in V_2$, then V is called the **orthogonal direct sum** of V_1 and V_2 .

- Proposition — Let V be a linear space. Then $V = V_1 \oplus V_2$ iff there is a linear operator $P : V \rightarrow V$ with $P^2 = P$ such that in the above decomposition $v_1 = Pv$ and $v_2 = (I - P)v$, and also $V_1 = P(V)$ and $V_2 = (I - P)(V)$.
- Note that $V = P(V) \oplus (I - P)(V)$

APPROXIMATION IN HILBERT SPACES II

- If V is a **Banach space**, P is called the **projection operator**
- If V is a **Hilbert space** and $V = P(V) \oplus (I - P)(V)$ is an **orthogonal direct sum**, the P is called the **orthogonal** projection operator

- It is easy to see that if a projection operator is **orthogonal**, then

$$(Pv, (I - P)w) = 0, \quad \forall v, w \in V$$

In other words, when $v = w$ we note that the best approximation in $P(V)$ of an element $v \in V$ and the approximation error are **orthogonal** !

- Example — Let $V = C[a, b]$, $V_1 = \mathcal{P}_n$ be the space of polynomials of degree less than or equal to n and let $\Delta : a \leq x_0 \leq x_1 \leq \dots \leq x_n \leq b$ be a partition of the interval $[a, b]$. For $v \in C[a, b]$, we define $Pv \in \mathcal{P}_n$ to be **the Lagrange interpolant** of v corresponding to the partition Δ , i.e., Pv satisfies the interpolation conditions $Pv(x_i) = v(x_i)$, $0 \leq i \leq n$. The interpolant Pv is uniquely determined, hence thus defined operator P is a **projector** (although not an orthogonal one). Analogous statements can be made for piecewise polynomial interpolation.

APPROXIMATION IN HILBERT SPACES III

- Let V_n be an n -dimensional subspace of a Hilbert space V . Suppose $\{u_1, \dots, u_n\}$ be an orthogonal basis of V_n . For any $v \in V$, the formula

$$Pv = \sum_{i=1}^n (u_i, v)u_i$$

defines an **orthogonal** projection from V onto V_n .

- How to find orthogonal systems in function (e.g., polynomial) spaces? — Perform **Schmidt orthogonalization procedure** on the system $\{1, x, \dots, x^n\}$
- Various families of **ORTHOGONAL POLYNOMIALS** are obtained depending on the choice of:
 - the domain $[a, b]$ over which the polynomials are defined, and
 - the weight w characterizing the inner product $(u, v) = \int_a^b wuv dx$ used for Schmidt orthogonalization

ORTHOGONAL POLYNOMIALS I

- Polynomials defined on the interval $[-1, 1]$
 - Legendre polynomials ($w = 1$)

$$P_k(x) = \sqrt{\frac{2k+1}{2}} \frac{1}{2^k k!} \frac{d^k}{dx^k} (x^2 - 1)^k, \quad k = 0, 1, 2, \dots$$

- Jacobi polynomials ($w = (1-x)^\alpha (1+x)^\beta$)

$$J_k^{(\alpha, \beta)}(x) = C_k (1-x)^{-\alpha} (1+x)^{-\beta} \frac{d^k}{dx^k} [(1-x)^{\alpha+k} (1+x)^{\beta+k}] \quad k = 0, 1, 2, \dots,$$

where C_k is a very complicated constant

- Chebyshev polynomials ($w = \frac{1}{\sqrt{1-x^2}}$)

$$T_n(x) = \cos(k \arccos(x)), \quad k = 0, 1, 2, \dots,$$

Note that Chebyshev polynomials are obtained from Jacobi polynomials for $\alpha = \beta = -1/2$

ORTHOGONAL POLYNOMIALS II

- Polynomials defined on the **periodic** interval $[-\pi, \pi]$
Trigonometric polynomials ($w = 1$)

$$S_k(x) = e^{ikx} \quad k = 0, 1, 2, \dots$$

- Polynomials defined on the interval $[0, +\infty]$
Laguerre polynomials ($w = e^{-x}$)

$$L_k(x) = \frac{1}{k!} e^x \frac{d^k}{dx^k} (e^{-x} x^k), \quad k = 0, 1, 2, \dots$$

- Polynomials defined on the interval $[-\infty, +\infty]$
Hermite polynomials ($w = 1$)

$$H_k(x) = \frac{(-1)^k}{(2^k k! \sqrt{\pi})^{1/2}} e^{x^2} \frac{d^k}{dx^k} e^{-x^2}, \quad k = 0, 1, 2, \dots$$

ORTHOGONAL POLYNOMIALS III

- Each of the aforementioned families of **orthogonal polynomials** forms the set of eigenvectors for the following **Sturm–Liouville problem**

$$\frac{d}{dx} \left[p(x) \frac{dy}{dx} \right] + [q(x) + \lambda r(x)] y = 0$$

$$a_1 y(a) + a_2 y'(a) = 0$$

$$b_1 y(b) + b_2 y'(b) = 0$$

for appropriately selected domain $[a, b]$ and coefficients p, q, r, a_1, a_2, b_1 and b_2 .

- As regards estimates of approximation errors in the different bases, consult
 - for trigonometric polynomials:
http://www.math.mcmaster.ca/~bprotas/MATH745/spectr_01.pdf,
pages 70–72
 - for Chebyshev polynomials:
http://www.math.mcmaster.ca/~bprotas/MATH745/spectr_03.pdf
page 111